Classical Knot Concordance
and
Homology Cobordism of 3-Manifolds

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Homology Cobordism of 3-Manifolds

Definition. Two 3-manifolds M_0 and M_1 are TOP homology cobordant if there is a compact topological 4-manifold W such that $\partial W = M_0 \coprod -M_1$ and both inclusion maps $M_i \to W$ induce isomorphisms on homology.

- TODAY: New Information about homology cobordism of 3-manifolds

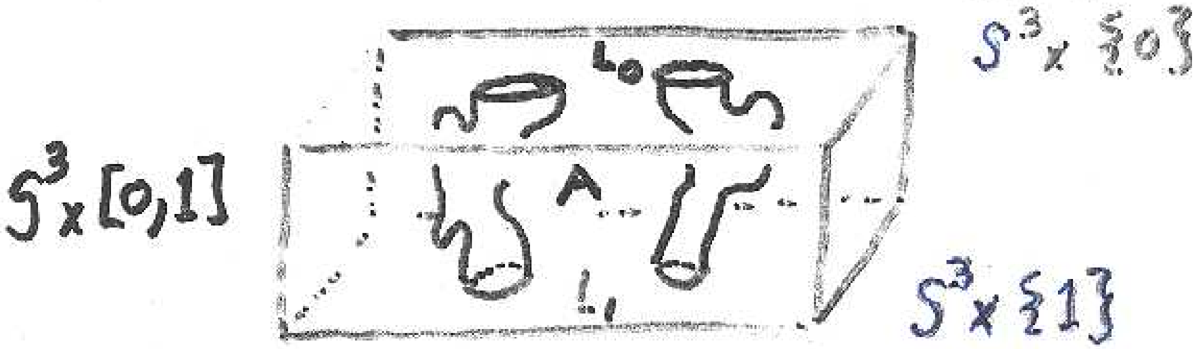
- Goal = Complete Classification

- π_1 may vary - HOW ??

- Every homology 3-sphere is TOP homology cobordant to S^3 (Freedman).
- any M is homology cobordant to hyperbolic M'
- Fundamental example for higher Betti number is Knot and Link Concordance

Link Concordance

Definition. Two m-component links L_0 and L_1 are (TOP) concordant if there is a disjoint collection A of m TOP (locally flatly) embedded annuli in $S^3 \times [0,1]$ such that $\partial A = L_0 \amalg L_1$.



- set of concordance classes of knots is abelian group $\mathcal{C}$ under connected sum

- L_i concordant $\Rightarrow$ Exteriors are TOP homology cobordant

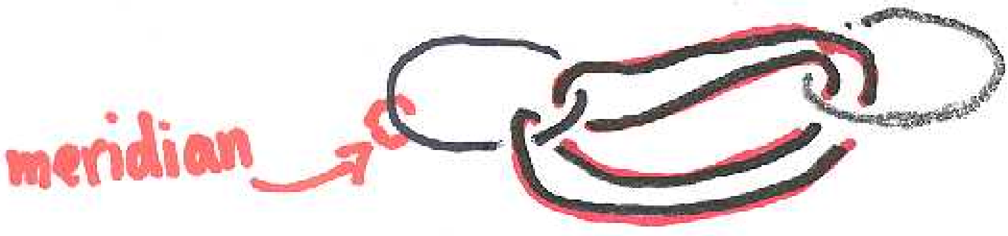
Concordance $\approx$ Homology cobordism of 3-manifolds

BUT...π_1 can vary

Classical Isotopy Invariants of Links and 3-Manifolds

- Alexander Module and Blanchfield form
- Signatures
- Arf Invariants
- Milnor's μ-invariants for links

Inductively determine if the meridional map $F \to G = \pi_1(S^3 - L)$ induces isomorphisms $F/F_n \to G/G_n$, where $G_n = [G_{n-1}, G]$ is lower central series. Equivalently, determine if longitudes can be written as n^{th}-order commutators.



Classical Alexander Module

L m-component link
$M = S^3 - L$
$\tilde{M}$ universal abelian cover with deck translations $\mathbf{Z}^m$

Definition. The classical Alexander Module is $H_1(\tilde{M}; \mathbf{Q})$ as a Λ-module, where $\Lambda = \mathbf{Q}[\mathbf{Z}^m] = \mathbf{Q}[x_1^{\pm 1}, ..., x_m^{\pm 1}]$; if $m = 1 \cong FREE \oplus TORSION$.

- r=rank of "FREE part"
- torsion submodule, for knot $\mathcal{A} \cong \Lambda / < p_1(t) > \oplus ... \oplus \Lambda / < p_k(t) >$
- Alexander polynomial $= \prod p_i(t)$
- Blanchfield form $\mathcal{B}l : \mathcal{A} \times \mathcal{A} \to \mathcal{K}(\Lambda)$ mod Λ.

These isotopy invariants behave well under concordance:

<u>Signatures</u> are concordance invariants if z is not a root of Alex. polynomial.

Eg:  $\not\equiv$ O

$\sigma_0 = 2$ $\qquad\qquad$ $\sigma_0 = 0$

<u>Arf invariant</u> is concordance invt.

Eg: Figure 8 $\not\equiv$ O

<u>Alexander</u> p<u>o</u>ly<u>nomial</u> of a knot concordant to unknot must be of form $f(t)f(t^{-1})$

<u>rank</u> of <u>Alexander</u> <u>module</u> of link is concordance invt

<u>Milnor's Invariants</u> are concordance invt

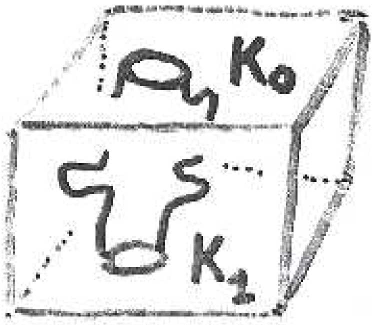
<u>Eg</u>:

 $\not\equiv$ O O

$\bar{\mu}(1122) = 1$ $\qquad\qquad$ $\bar{\mu}(1122) = 0$

Why do these behave well under concordance ?

Alexander Module, Blanchfield form, Signatures, Arf Invariant:
Because they are defined via the abelianization map $\phi : \pi_1(S^3 - K) \to H_1(S^3 - K) \cong \mathbf{Z}$ and so ϕ extends over a homology cobordism.



$$\begin{tikzcd}
\pi_1(S^3 - K_0) \arrow[d] \arrow[dr, two heads] & \\
\pi_1(S^3 \times I - C) \arrow[r, two heads] & \mathbf{Z} \\
\pi_1(S^3 - K_1) \arrow[u] \arrow[ur, two heads] &
\end{tikzcd}$$

Similarly for Milnor's invariants for links due to:

Theorem. (J. Stallings 1963) If $f : X \to Y$ is a homology equivalence and $A = \pi_1(X)$ and $B = \pi_1(Y)$ then f induces isomorphisms $A/A_n \cong B/B_n$ for every n.

$A = \pi_1(S^3 \setminus L_0)$

A/A_n

$\cong C/C_n$

$B/B_n \cong$

$B = \pi_1(S^3 \setminus L_1)$

BEYOND CLASSICAL INVARIANTS

There are <u>many</u> <u>higher-order</u> invariants but they are <u>NOT</u> <u>concordance</u> invariants:

- take $\mathbb{Z} \times \{\text{finite group}\}$ covering space – twisted Alexander polynomials (Lin, Casson-Gordon, Kirk-Livingston, Cha, Turaev, Kim-Friedl.
- more generally, associated to <u>any</u> $\phi : \pi_1(M^3) = G \longrightarrow \Gamma$ are
    - higher-order Alexander modules
    - " " Blanchfield linking form
    - signatures
    - "Arf" invariants

I will explain some of these, but they are not concordance invariants.

Higher-Order Alexander Modules

Take iterated universal abelian covering spaces

$$\begin{array}{ll}
\widetilde{M}_n & \pi_1 = G^{(n+1)} = [G^{(n)}, G^{(n+1)}] \\
\downarrow & \\
\vdots & \\
\downarrow & \\
\widetilde{M}_0 & \pi_1 = G^{(1)} = [G, G] \\
\downarrow & \\
M & \pi_1(M) = G
\end{array}$$

(n+1)-st term of derived series

Definition: $\mathcal{A}_n$, n^{th} order Alexander Module

is $H_1(\widetilde{M}_n) = G^{(n+1)}/G^{(n+2)}$ as a module

over $\mathbb{Z}\Gamma_n = \mathbb{Z}[G/G^{(n+1)}]$

More generally for any $\phi : \pi_1(M) \longrightarrow \Gamma$ consider the covering space induced

$$\begin{array}{c} \tilde{M}_\Gamma \\ \big\downarrow \\ M \end{array} \qquad \mathcal{A}_\Gamma(M) \equiv H_1(\tilde{M}_\Gamma) \text{ as } \mathbb{Z}\Gamma\text{-module}$$

"higher-order Alexander module"

can also be defined using homology with coëfficients:

$$H_1\Big(C(\tilde{M}_{\substack{\text{univeral}\\ \text{cover}}}) \underset{\mathbb{Z}\pi_1(M)}{\otimes} \mathbb{Z}\Gamma\Big)$$

$$\Vert \text{ notation}$$

$$H_1(M; \mathbb{Z}\Gamma)$$

$$\cong$$

$$\mathcal{A}_\Gamma(M)$$

Working with modules over noncommutative rings $\mathbb{Z}\Gamma$ is difficult.

Some tools:

- If Γ is a torsion-free-solvable group such as is obtained from iterated extensions of torsion-free-abelian groups (Poly-Torsion-Free-Abelian) (PTFA) then $\mathbb{Z}\Gamma$ is an integral domain and has a "field of fractions"

$$\mathbb{Z}\Gamma \hookrightarrow \mathcal{K}\Gamma = \left\{ rs^{-1} \,\middle|\, \begin{array}{l} r, s \in \mathbb{Z}\Gamma \\ s \neq 0 \end{array} \right\}$$

Some FACTS

- any $\mathcal{K}$ module is a $\mathcal{K}$-vector space $\cong \mathcal{K}^m$
- for any $\mathbb{Z}\Gamma$-module, $\mathcal{A}$, the Kernel of $\mathcal{A} \longrightarrow \mathcal{A} \otimes_{\mathbb{Z}\Gamma} \mathcal{K}\Gamma$ is the torsion submodule of $\mathcal{A}$

Lemma: The higher Alexander modules of any space with $\beta_1 = 1$ are Torsion-modules

Higher-order Blanchfield Linking FORMS

Given $\phi : \pi_1(M) \longrightarrow \Gamma$ $M = 3$-manifold

$$Bl_\Gamma : \mathcal{A}_\Gamma \longrightarrow \operatorname{Hom}_{\mathbb{Z}\Gamma}\left(\mathcal{A}_\Gamma, \mathcal{K}\Gamma/\mathbb{Z}\Gamma\right)$$

defined as follows:

$$\begin{tikzcd}
\mathcal{A}_\Gamma \arrow[r, phantom, "\equiv"] \arrow[dddd, dashed, "Bl_\Gamma"'] & H_1(M;\mathbb{Z}\Gamma) \arrow[d] \\
 & H_1(M,\partial M;\mathbb{Z}\Gamma) \arrow[d, "\text{Poincaré Duality}", "\cong"'] \\
 & H^2(M;\mathbb{Z}\Gamma) \\
 & H^1(M;\mathcal{K}/\mathbb{Z}\Gamma) \arrow[u, "\text{Bockstein}", "\cong"'] \arrow[d, "\kappa \text{ evaluation}"] \\
\operatorname{Hom}(\mathcal{A}_\Gamma, \mathcal{K}\Gamma/\mathbb{Z}\Gamma) \arrow[r, phantom, "\equiv"] & \operatorname{Hom}(H_1(M), \mathcal{K}\Gamma/\mathbb{Z}\Gamma)
\end{tikzcd}$$

cohomology Bockstein sequence for

$$0 \to \mathbb{Z}\Gamma \xrightarrow{i} \mathcal{K}\Gamma \xrightarrow{\pi} \mathcal{K}\Gamma/\mathbb{Z}\Gamma \to 0$$

is

$$\underset{\substack{\| \\ 0}}{H^1(M;\mathcal{K})} \xrightarrow{\pi} H^1(M;\mathcal{K}/\mathbb{Z}\Gamma) \xrightarrow[\cong]{\delta} H^2(M;\mathbb{Z}\Gamma) \xrightarrow{i} \underset{\substack{\| \\ 0}}{H^2(M;\mathcal{K})}$$

these modules are 0 because one can show higher-order Alexander Modules of Knots, $\mathcal{A}_{0_\Gamma} = H_1(M;\mathbb{Z}\Gamma)$, are TORSION MODULES so $H_1(M;\mathcal{K}\Gamma) \cong H_1(M;\mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathcal{K} = 0$

$$\Rightarrow H_*(M;\mathcal{K}) \cong H^*(M;\mathcal{K}) \cong 0$$

These higher-order modules and forms are useful invariants of 3-manifolds

- higher-order Alexander module $\mathcal{A}_\Gamma = H_1(M_\Gamma)$ module over noncommutative ring $\mathbb{Z}\Gamma$ (Orr-Teichner-C). Main example: $\mathcal{A}_n = G^{(n)}/G^{(n+1)}$.

Higher-order modules of a knot K are nontrivial unless K has Alexander polynomial 1 (Cochran)

- give new obstructions to fibering (C, Harvey)

- degree of higher polynomials give lower bounds for Thurston norm (S. Harvey)

- lead to new obstructions to symplectic structure on $M \times S^1$ (Harvey)

- Higher-order Blanchfield forms stronger than module alone (C.Leidy)

- higher-order Arf Invariants: Given any $\phi : \pi_1(M) = G \to \Gamma$ we can consider the fundamental class of the zero surgery M on a knot K:

$$([M],\phi) \qquad [M] \in \Omega_3^{Spin}(K(\Gamma,1))$$

- generalizes classical Arf invariant

- distinct from higher-order modules (C)

- can obstruct being a ribbon knot (Orr-Teichner-C, Orr-C)

- Higher-Order Signatures: Given any $\phi : \pi_1(M) = G \to \Gamma$ we can consider the Cheeger-Gromov-Atiyah-von Neumann ρ invariant $\rho_\Gamma = \rho(M,\phi)$ a real number.

Why do these behave well under homology cobordism?

ANSWER: They do not.

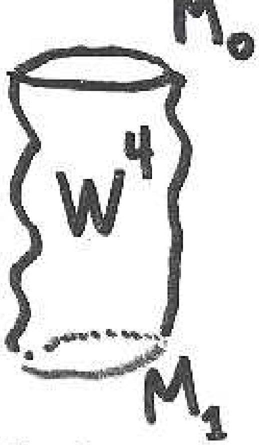
MORAL(again) We have some nice good subtle invariants that depend on π_1, but to use them for the problem of homology cobordism we must have better answers to:

Question(again): In a homology equivalence or a homology cobordism of 3-manifolds, what aspects of π_1 are preserved besides the abelianization H_1?

~~I now mention~~ one [redacted] ~~recent theorems giving~~ [redacted] new [redacted] ~~answers to this question, which forms the key to the proofs of~~ [redacted] ~~new theorems about knot~~ [redacted] ~~concordance.~~ ~~Later Shelly Harvey will discuss another answer that helps study links.~~

MORAL: To get beyond classical invariants we must have a better answer to:

Question: In a homology cobordism of 3-manifolds how does π_1 vary ? What aspects of π_1 are preserved?



$$\begin{tikzcd} \pi_1(M_0) \arrow[d, "i_*"] \arrow[dr, "\phi_0"] & \\ \pi_1(W) \arrow[r, "\psi"] & ? = \Gamma \\ \pi_1(M_1) \arrow[u] \arrow[ur, "\phi_1"'] & \end{tikzcd}$$

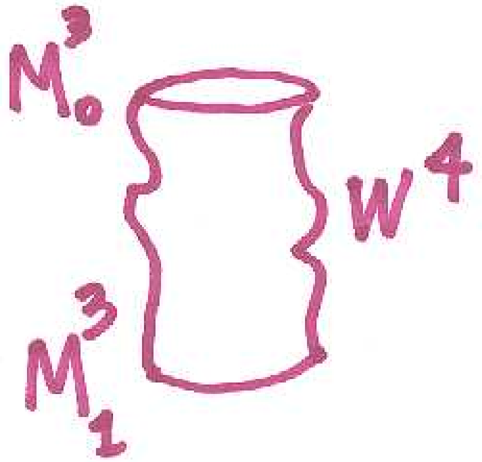
What groups Γ can we put above?

- This talk will give new answer for knots.
- In next talk by Shelly Harvey, will see new answer for links + groups in general
- Second talk by Shelly Harvey = Applications to 3-manifolds
- Second talk by Cochran = Applications to Knot concordance and other

The Problem

Suppose K_0, K_1 are concordant knots so that M_0, M_1 (the 0-framed Dehn surgeries on the knots) are homology cobordant 3-mfds.

What is true about π_1 of the cobordism?



$$
\begin{tikzcd}
\pi_1(M_0) \arrow[d, "i_*"] \arrow[r, dashed, "?"] & \Gamma \ ? \\
\pi_1(W) \arrow[ur, dashed, "?"']
\end{tikzcd}
$$

If, for example, $\pi_1(W) \cong \mathbb{Z}$ then the only group Γ that works is $\Gamma = \mathbb{Z}$ and so the classical invariants associated to the $\mathbb{Z}$ cover would be the only invariants.

Suppose K_0 and K_1 are concordant knots so M_0, M_1 are homology cobordant via W^4.

$$\begin{tikzcd}
\pi_1(M_0) \arrow[d, "i_*"] \arrow[dr, dashed] & \\
\pi_1(W) \arrow[r] & \pi_1(W)/\pi_1(W)^{(n)} \equiv \Gamma
\end{tikzcd}$$

since Γ is a coefficient system for M_0 AND W

$$\mathcal{A}_\Gamma(M_0) \cong H_1(M_0;\mathbb{Z}\Gamma),\ \mathcal{A}_\Gamma(W) \cong H_1(W;\mathbb{Z}\Gamma)$$

are defined and

<u>Theorem</u> (Taehee Kim - C) "Non-triviality"
Let $d =$ degree $\Delta_{K_0}(t)$. Then for $i: M_0 \to W$

$$i_*: \mathcal{A}_\Gamma(M_0) \to \mathcal{A}_\Gamma(W)$$

"size" image $i_* \geq \frac{1}{2}$ "size" $\mathcal{A}_\Gamma(M_0) \geq \frac{d-2}{2}$

In particular if $d \geq 4$ then the image of i_* is non-trivial.

But notice that

$$\mathcal{A}_{\rho}(W) = H_1\left(W; \mathbb{Z}\left[\frac{\pi_1(W)}{\pi_1(W)^{(n)}}\right]\right) \cong \frac{\pi_1(W)^{(n)}}{\pi_1(W)^{(n+1)}}$$

<u>Corollaries</u>: A If K_0 is concordant to K_1 and $\deg \Delta_{K_0} \geq 4$ then the fundamental group of the complement of the concordance is <u>not</u> <u>solvable</u>.

B In the above situation:

$$\frac{\pi_1(M_0)^{(n)}}{\pi_1(M_0)^{(n+1)}} \xrightarrow{i_*} \frac{\pi_1(W)^{(n)}}{\pi_1(W)^{(n+1)}}$$

has non-zero image $\forall n$.

Outline of Proof:

Step 1: Since W is homology cobordism $H_*(W, M_0; \mathbb{Z}) = 0$. Using homological algebra one shows $H_2(W, M_0; \mathbb{Z}\Gamma)$ is Torsion module.

Step 2: consider long exact homology sequence of pair (W, M_0) with $\mathbb{Z}\Gamma$ coeffs.

$$H_2(W, M_0) \xrightarrow{\partial_*} H_1(M_0) \xrightarrow{i_*} H_1(W)$$

Establish that ∂_* is "dual" to i_*

then $\ker i_* \cong \text{image } \partial_*$

$\cong$ image of "dual of ∂_*"

$\cong$ image i_*

$\Rightarrow$ image i_* has "size" $= \frac{1}{2}$ "size" $H_1(M_0)$

$$
\begin{tikzcd}
H_2(W, M_0) \arrow[r, "\partial_*"] \arrow[d] & H_1(M_0) \arrow[r, "i_*"] \arrow[dd, "\text{P.D.}", "\cong"'] \arrow[dddd, bend left=70, "Bl_\Gamma"] & H_1(W) \\
H_2(W, \partial W) \arrow[d, "\text{P.D.}", "\cong"'] & & \\
H^2(W) \arrow[r, "i^*"] & H^2(M_0) & \\
H^1(W; \mathcal{K}/\mathbb{Z}\Gamma) \arrow[u, "\text{Bockstein}"'] \arrow[r, "i^*"] \arrow[d] & H^1(M_0; \mathcal{K}/\mathbb{Z}\Gamma) \arrow[u, "\delta^*\ \text{Bockstein}"', "\cong"] \arrow[d, "\kappa", "\cong"'] & \\
\mathrm{Hom}(H_1(W), \mathcal{K}/\mathbb{Z}\Gamma) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(H_1(M_0), \mathcal{K}/\mathbb{Z}\Gamma) \arrow[d, equal] & \\
\mathrm{Hom}(\mathcal{A}_\Gamma(W); \mathcal{K}/\mathbb{Z}\Gamma) \arrow[r, "i^*"] & \mathrm{Hom}(\mathcal{A}_\Gamma(M_0), \mathcal{K}/\mathbb{Z}\Gamma) \arrow[d, equal] & \\
\mathcal{A}_\Gamma(W)^* \arrow[r, "\text{dual of } i_*"] & \mathcal{A}_\Gamma(M_0)^* &
\end{tikzcd}
$$

$\beta_1(M_0) = 1$

Summary:

Given a (topological) homology cobordism between three dimensional manifolds, we have found severe restrictions on $\pi_1(W^4)$ and on how it relates to $\pi_1(M_0^3)$ and $\pi_1(M_1^3)$. It cannot be "too small".

If the classical Alexander polynomial of M_0 has degree ≥ 4 then $\pi_1(W)$ is not virtually solvable and

$$\frac{\pi_1(M_0)^{(n)}}{\pi_1(M_0)^{(n+1)}} \xrightarrow{i_*} \frac{\pi_1(W)^{(n)}}{\pi_1(W)^{(n+1)}}$$

is non-trivial $\forall n \geq 0$