Classical Knot Concordance and Homology Cobordism of 3-manifolds

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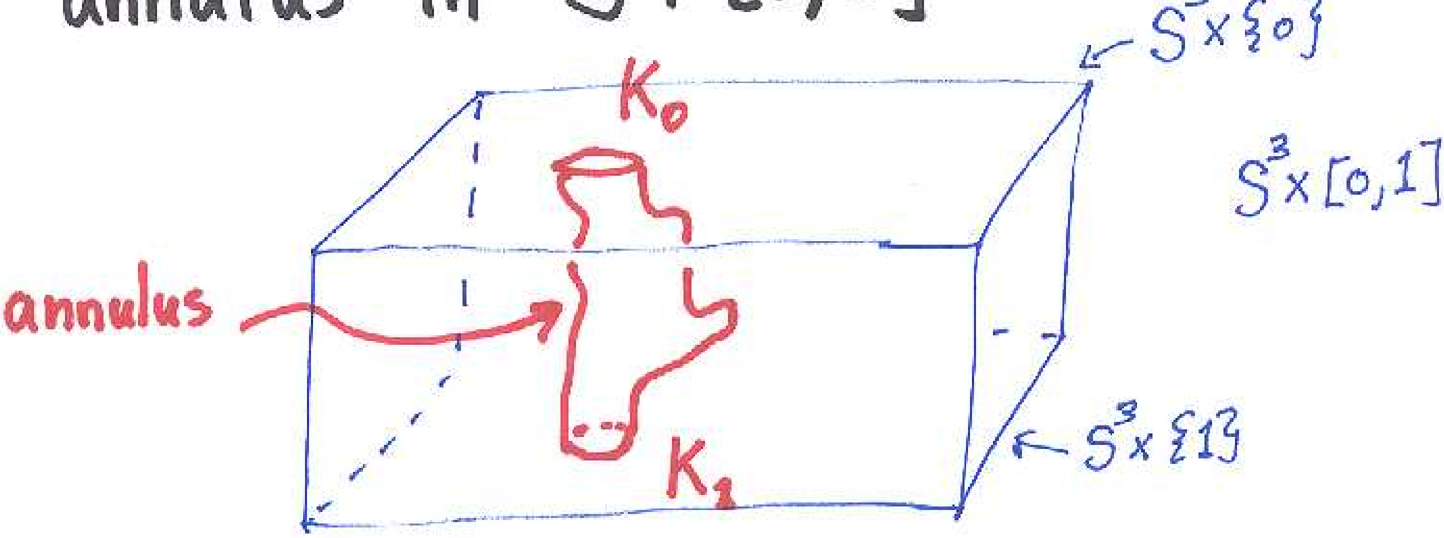
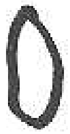
GOAL: Understand when does a knot in S^3 bound an embedded disk in B^4 (such a knot is called a <u>slice</u> knot)

more generally, when are knots K_0, K_1 <u>concordant</u>, i.e. they co-bound an annulus in $S^3 \times [0,1]$



to study this we "filter" the set of all knots by studying approximations to disks and annuli called <u>gropes</u>

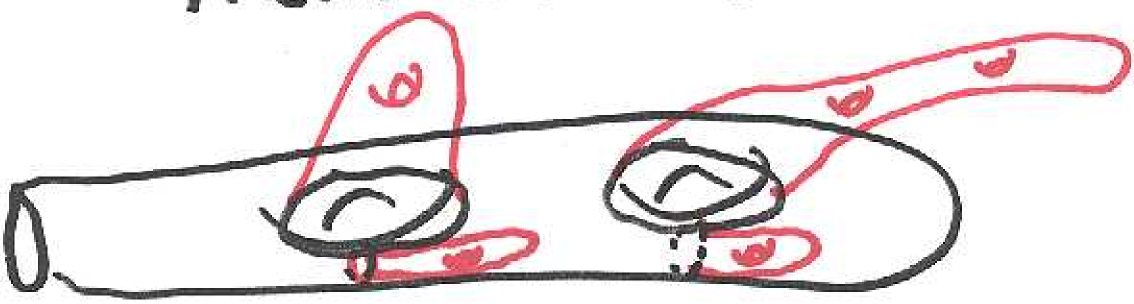
Gropes

 = Grope of Height 0

 = Grope of Height 1

OR

To form a Grope of Height 2 consider a "symplectic basis" of circles on a Height one grope and have each of them bound surfaces:



note: γ bounds non-embedded n-grope

$\Longleftrightarrow [\gamma] \in G^{(n)}$

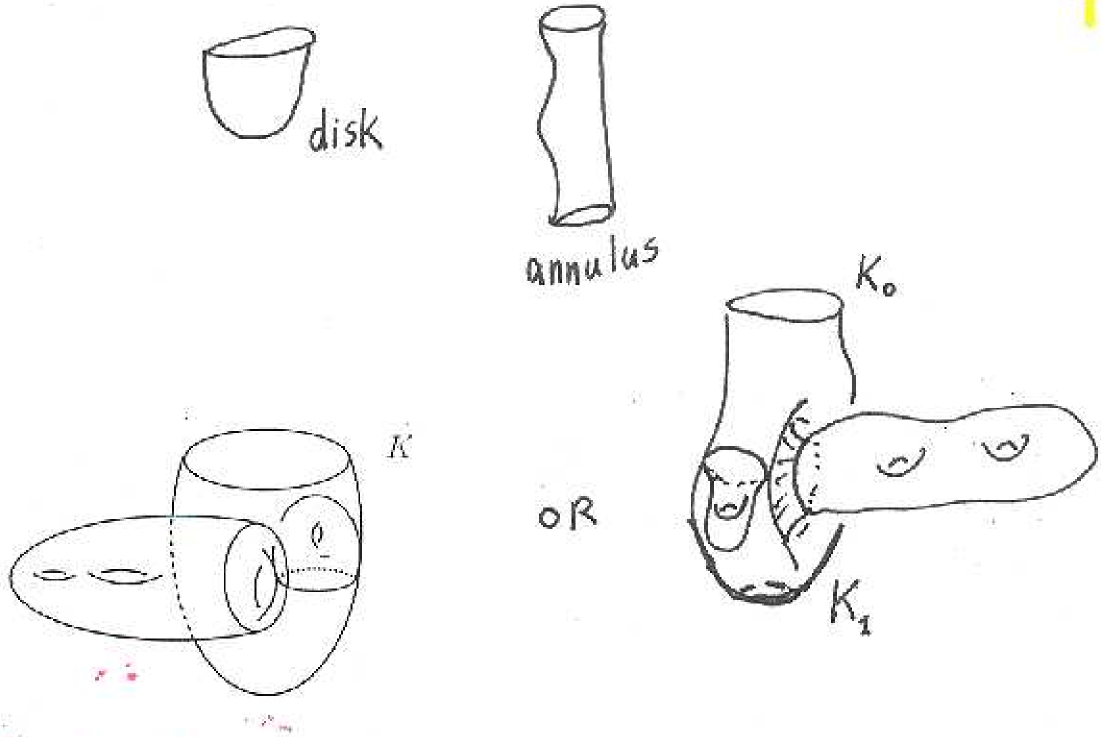
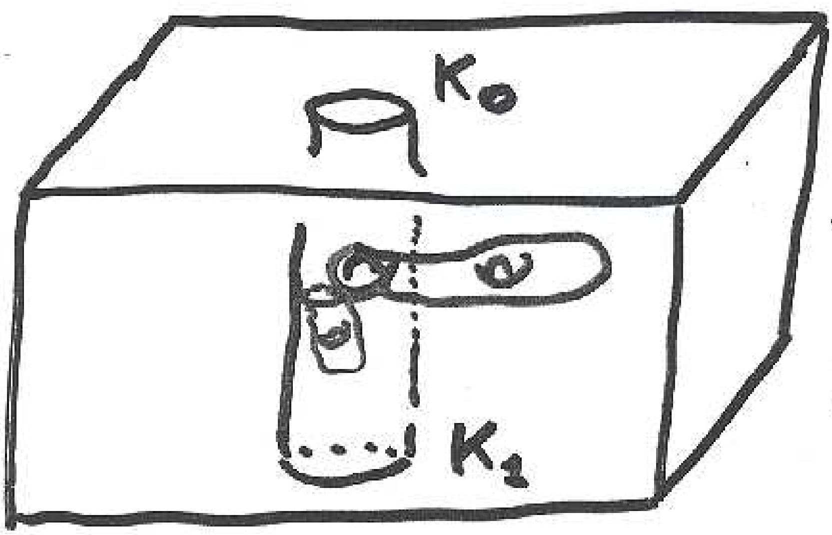


FIGURE 1. A Grope of height 2.

Def: Two knots are n-grope concordant if they cobound in $S^3 \times I$ an n-grope.



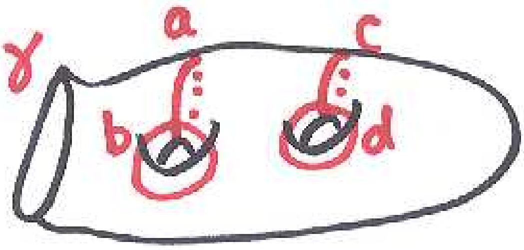
$S^3 \times [0,1]$

These gropes are related to group theory:

- a circle $\gamma \looparrowright X$ bounds a map of an orientable surface  $\looparrowright X$

$$\Updownarrow$$

$$[\gamma] \in [\pi_1(X), \pi_1(X)]$$

since 

$[\gamma] = [a,b][c,d]$

<u>Easy</u> <u>Lemma</u>: a circle $\gamma \looparrowright X$ bounds a <u>mapped</u>-<u>in</u> height n grope iff

$$[\gamma] \in \pi_1(X)^{(n)}$$

But here today we require <u>embedded</u> gropes. All of Knot theory is the difference between embedded and <u>not</u> embedded since any circle in S^3 is null-homotopic

- other families of gropes suggested by other series, eg. lower central series
  - related to Kontsevitch Integral (gropes in S^3)
- ∞ height gropes - Freedman
  Casson
-

- concordant Knots are n-grope concordant for every n

- a Knot is n-grope concordant to the trivial Knot if and only if K bounds a height n-grope in B^4.

<u>Define</u>: $\mathcal{G}_n = \{$Knots that bound embedded height n-gropes in $B^4\}$

This gives a filtration of all Knots:

$$\{\text{TRIVIAL KNOT}\} \ldots\ldots \subseteq \mathcal{G}_n \subseteq \ldots \subseteq \mathcal{G}_2 \subseteq \mathcal{G}_1 = \text{ALL KNOTS}$$

since $\mathcal{G}_n$ is closed under connected-sum and concordance it can be viewed as a filtration of the classical Knot concordance group $\mathcal{C}$

$$\{\text{all slice Knots}\} \subseteq \ldots\ldots \subseteq \mathcal{G}_n \subseteq \ldots \subseteq \mathcal{G}_2 \subseteq \mathcal{C} = \text{ALL KNOTS}$$

What was previously Known:

$$\mathcal{G}_5 \subseteq \mathcal{G}_4 \subseteq \mathcal{G}_3 \subseteq \mathcal{G}_2 \subseteq \mathcal{G}_1 = \mathcal{C}$$

non-trivial

$\mathcal{G}_1/\mathcal{G}_2 \cong \mathbb{Z}_2$ Arf Invariant

$\mathcal{G}_2/\mathcal{G}_3 \cong \mathbb{Z}^\infty \times \mathbb{Z}_2^\infty \times \mathbb{Z}_4^\infty$ Levine 1960's

$\mathcal{G}_3/\mathcal{G}_4 =$ infinite rank, infinite 2-torsion — Casson-Gordon, 70's; Jiang, Livingston, 90's; Kim, Friedl, 2002

$\mathcal{G}_4/\mathcal{G}_5 =$ infinite rank — Cochran-Orr-Teichner 2000

Today:

Thm (C-Teichner) $\forall n$ $\mathcal{G}_n / \mathcal{G}_{n+1}$ is infinite.

Thm (C-T. Kim) For any n and any knot K whose Alexander polynomial has degree at least 4, there are an ∞ number of knots that are n-grope concordant to K but are all distinct modulo $(n+1)$-grope concordance.

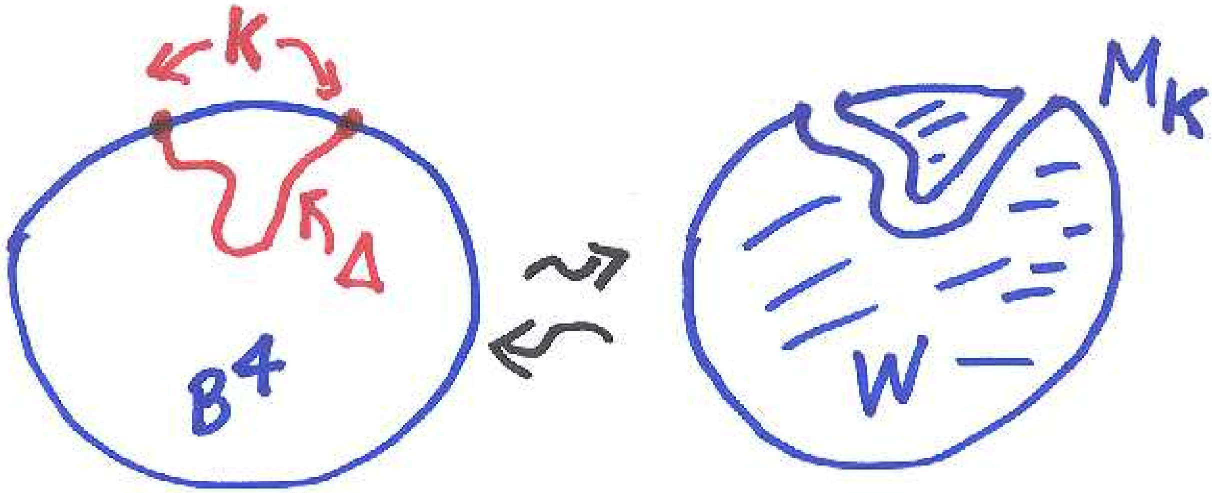
$\Rightarrow$ highly non-trivial structure in knot concordance group.

To prove the Theorems we first try to find an invariant that is zero for knots in $\mathcal{G}_n$.

Proposition: K is a slice knot in a homology B^4 $\Longleftrightarrow$ the 0-framed surgery M^3_K bounds a compact 4-manifold W^4 such that

1) $H_1(M) \xrightarrow{\cong} H_1(W) \cong \mathbb{Z}$
2) $H_2(W) \cong 0$

Proof:



Proposition: If $K \in \mathcal{G}_{n+2}$ then $M^3_K = \partial W^4$ such that

1. $H_1(M) \to H_1(W) \cong \mathbb{Z}$
2. $H_2(W) \cong \mathbb{Z}^{2r} = \langle l_i, d_i \mid i = 0, \ldots, r \rangle$

l_i, d_i are represented by embedded surfaces ~~such that~~ L_i, D_i

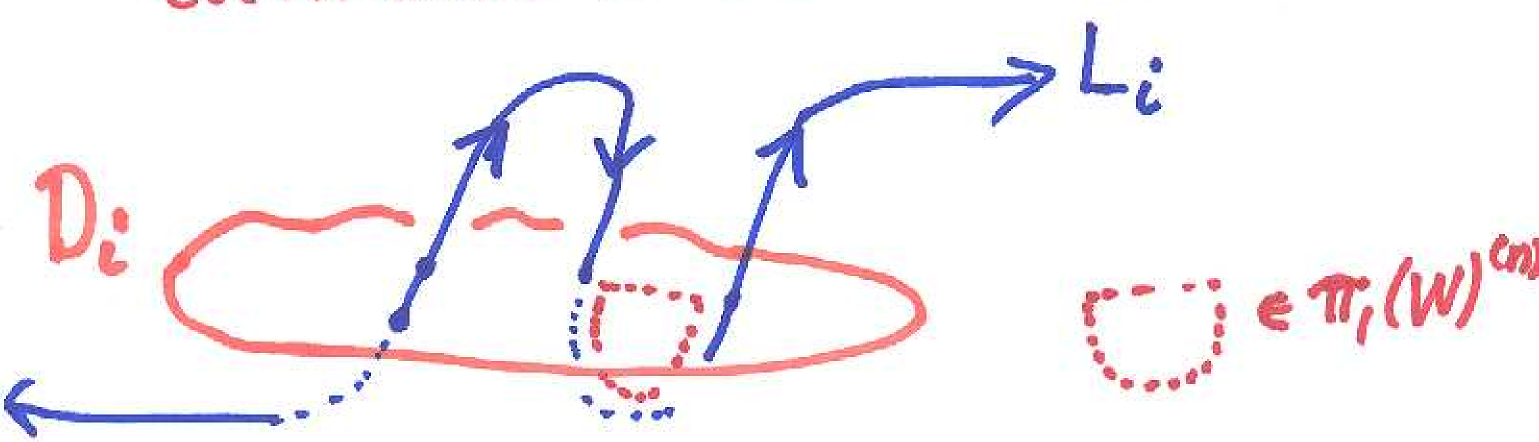
a. $\pi_1(L_i) \subseteq \pi_1(W)^{(n)}$

$\pi_1(D_i) \subseteq \pi_1(W)^{(n)}$

b. $L_i \cap L_j = 0 \quad D_i \cap D_j = 0$

$L_i \cap D_j = \delta_{ij}$

algebraic intersection numbers with coefficients in $\mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}]$



This means that if we consider $\widetilde{W}_n$, the covering space of W with $\pi_1(\widetilde{W}_n) = \pi_1(W)^{(n)}$, then

a. $\dfrac{H_2(\widetilde{W}_n)}{\text{Torsion}}$ as a $\mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}]$ has a basis given by the lifts $\widetilde{L}_i, \widetilde{D}_i$

b. with respect to this basis the equivariant intersection form on

$$H_2(\widetilde{W}_n)/\text{Torsion} \times H_2(\widetilde{W}_n)/\text{Torsion} \to \mathbb{Z}\left[\frac{\pi_1(W)}{\pi_1(W)^{(n)}}\right]$$

has matrix

$$\begin{array}{c} \overbrace{\qquad\quad}^{r} \qquad\qquad\quad \\ \left(\begin{array}{ccc|ccc} 0&0&0&&& \\ 0&0&0&&*& \\ 0&0&0&&& \\ \hline &&&0&0&0 \\ &*&&0&0&0 \\ &&&0&0&0 \end{array}\right) \end{array} = \bigoplus_{1}^{r} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This suggests we use "signature" of $H_2(\widetilde{W}_n)$ as an invariant.

Cheeger-Gromov von Neumann ρ-invariant

Given any regular covering space of a 3-manifold M^3

$$\begin{array}{c} M_\Gamma \\ \downarrow \\ M \end{array}$$

(equivalently to any $\phi : \pi_1(M^3) \to \Gamma$)

$\underline{\rho(M,\phi) \in \mathbb{R}}$

In case $M^3 = \partial W^4$ and ϕ extends to $\psi : \pi_1(W^4) \longrightarrow \Gamma$, giving a Γ-cover of W, $\widetilde{W}_\Gamma$ then

$$\rho(M,\phi) = \sigma^{(2)}_\Gamma(\widetilde{W}_\Gamma) - \sigma(W)$$

von Neuman signature of equivariant intersection form on $H_2(\widetilde{W}_\Gamma)/\text{Torsion}$

usual signature of W, i.e. of intersection form on $H_2(W;\mathbb{Z})$

The equivariant intersection form:

$$\frac{H_2(\widetilde{W}_r)}{\text{Torsion}} \times \frac{H_2(\widetilde{W}_r)}{\text{Torsion}} \longrightarrow \mathbb{Z}\Gamma$$

is a Hermitian matrix whose entries lie in ring

so what do we mean by signature?

Using von Neumann algebras:

$$\text{Herm}_{n\times n}(\mathbb{Z}\Gamma) \longrightarrow \text{Herm}_{n\times n}(N\Gamma) \xrightarrow{\sigma_\Gamma^{(2)}} \mathbb{R}$$

"group von Neumann algebra of Γ"

Now suppose $K \in \mathcal{G}_{n+3}$ so $M_K = \partial W^4$.

Let $\Gamma = \pi_1(W)/\pi_1(W)^{(n+1)}$. Then we have

$$\begin{tikzcd}
\pi_1(M) \arrow[dr, "i_*"'] \arrow[rr, dashed, "\phi"] & & \Gamma = \pi_1(W)/\pi_1(W)^{(n+1)} \\
& \pi_1(W) \arrow[ur] &
\end{tikzcd}$$

<u>Thm</u>: If $K \in \mathcal{G}_{n+3}$, $\rho(M,\phi) = 0$

<u>Proof</u>: $\rho(M,\phi) \equiv \sigma^{(2)}_{\Gamma}(\widetilde{W}_{\Gamma}) - \sigma(W)$

but by previous proposition both the <u>ordinary and equivariant</u> intersection forms look like $\oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so have signature 0.

Thm: For any knot K where degree $\Delta_K \geq 4$ and any n there are knots $n+2$-grope concordant to K but not $(n+3)$-grope concordant to K.

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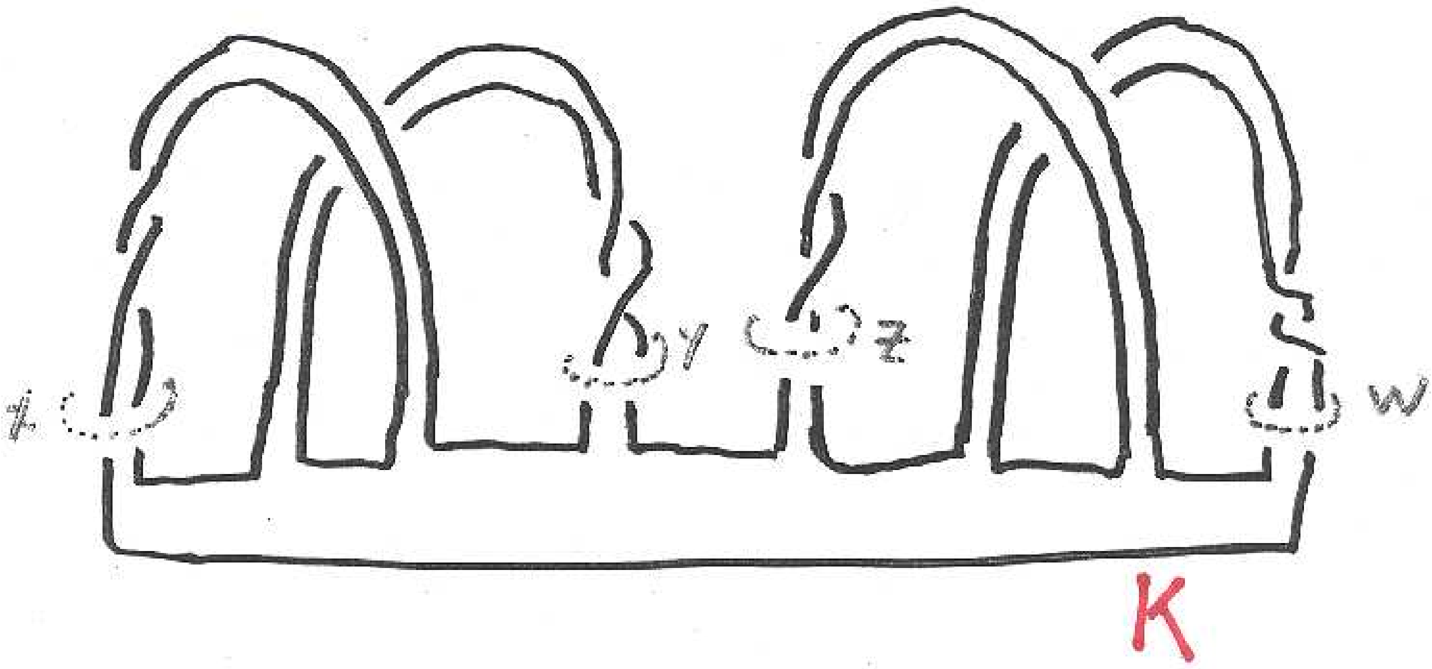
Proof of ~~First~~ Theorem

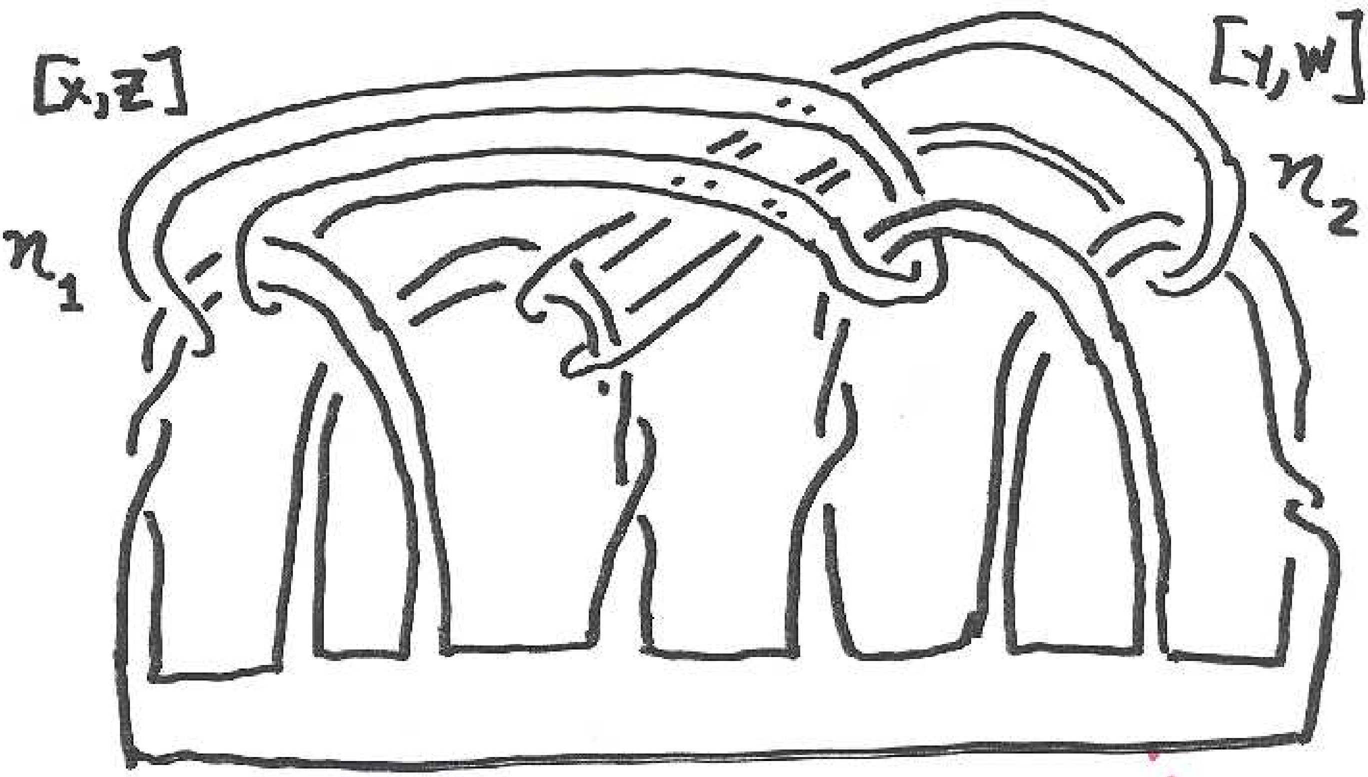
Start with knot K:



Find a very large number 2C:

Theorem. (J.Cheeger-M.Gromov) If M_K is the zero surgered 3-manifold, there is a **universal** bound $|\rho(M_K, \phi)| < C$ for **any** ϕ.

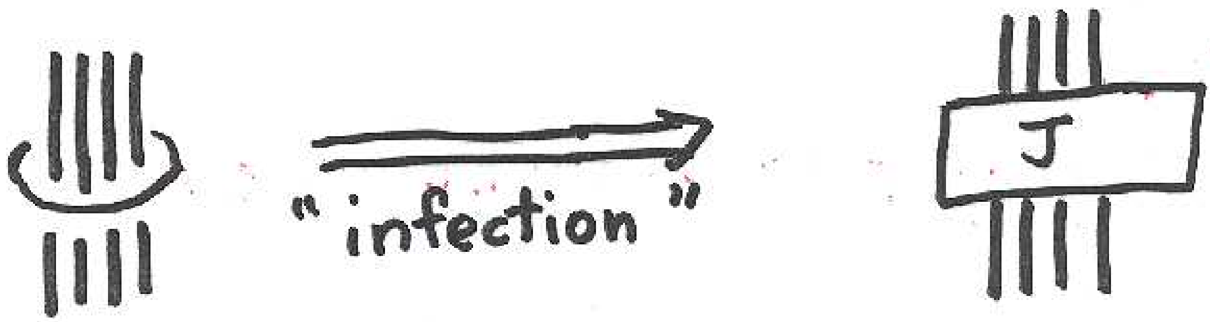
Choose circles $\eta_1, \ldots\ldots\ldots$ representing all the simple $(n-1)$ commutators in x, y, z, w. Since $x, y, z, w \in$ commutator subgroup, $\eta_i \in \pi_1(S^3 - K)^{(n)}$



Example: $n = 2$
not all η_i shown

Choose η_i to form unlink.
Moreover choose η_i to bound embedded n-gropes in $S^3 \setminus K$ (all disjoint).

For each η_i:



for some Knot J whose Levine-Tristram signature $|LT(J)| > 2C$

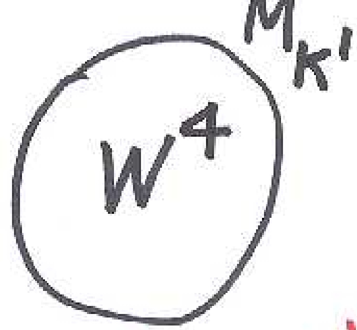
(for specific J use connected sum of trefoils)

Claim: resulting K' is $(n+2)$-grope concordant to K but not $(n+3)$-grope concordant to K.

Now show K' is NOT $(n+3)$-grope cobordant to K. For simplicity we just show when K is a slice knot so we just need to show $K' \notin \mathcal{G}_{n+3}$. By contradiction:

Suppose $K' \in \mathcal{G}_{n+3}$,

letting $\Gamma = \pi_1(W) / \pi_1(W)^{(n+1)}$



we have previously shown $\rho(M_{K'}, \phi') = 0$

where $\phi' : \pi_1(M_{K'}) \xrightarrow{i_*} \Gamma$.

However, since K' was obtained from K by slight modifications using J it is not hard to calculate:

$$\underbrace{\rho(M_{K'}, \phi')}_{\substack{\| \\ 0 \text{ (above)}}} = \underbrace{\rho(M_K, \phi)}_{\substack{\uparrow \\ \text{less than } C}} + \underbrace{\sum_{\text{infections}} \varepsilon_i \, LT(J)}_{\text{greater than } 2C}$$

contradiction AS LONG AS SOME $\varepsilon_i \neq 0$

$$\varepsilon_i = \begin{cases} 1 & \phi(\eta_i) \neq 0 \\ 0 & \phi(\eta_i) = 0 \end{cases}$$

$$\phi : \pi_1(M_K) \longrightarrow \pi_1(W) / \pi_1(W)^{(n+1)} \cong \Gamma$$

But the ~~"[illegible]~~ "non-triviality" Theorem from my first talk said ~~we could insure~~

$$\eta_i \in \frac{\pi_1(M_K)^{(n)}}{\pi_1(M_K)^{(n+1)}} \xrightarrow{\phi} \frac{\pi_1(W)^{(n)}}{\pi_1(W)^{(n+1)}}$$

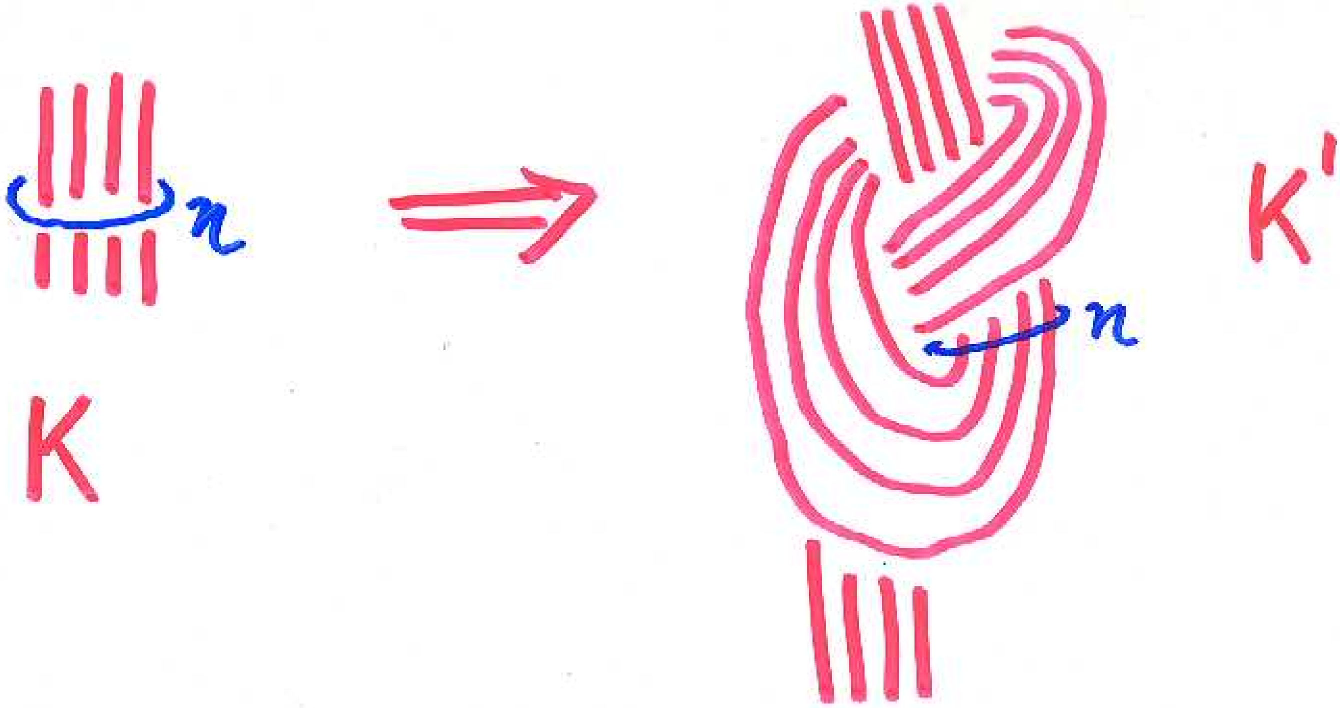
has non-zero image so if we choose $\eta_1, \ldots, \eta_K$ to generate the first module then at least one maps to non-zero!!

Now show other implication of

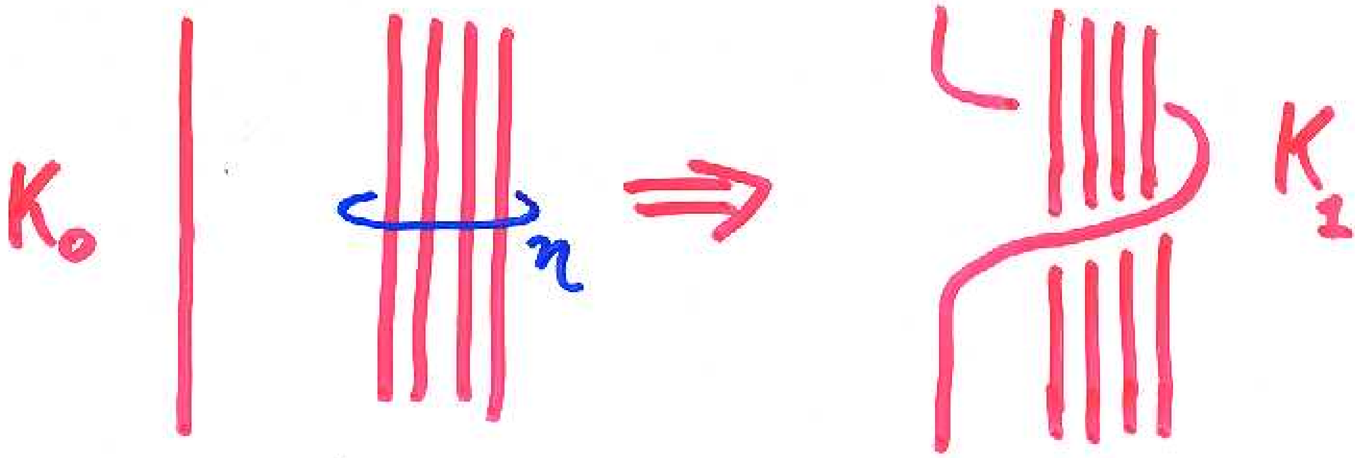
Theorem: K and K' are $(n+2)$-grope concordant.

I will show easy proof that they are n-grope concordant.

The full result is a little harder

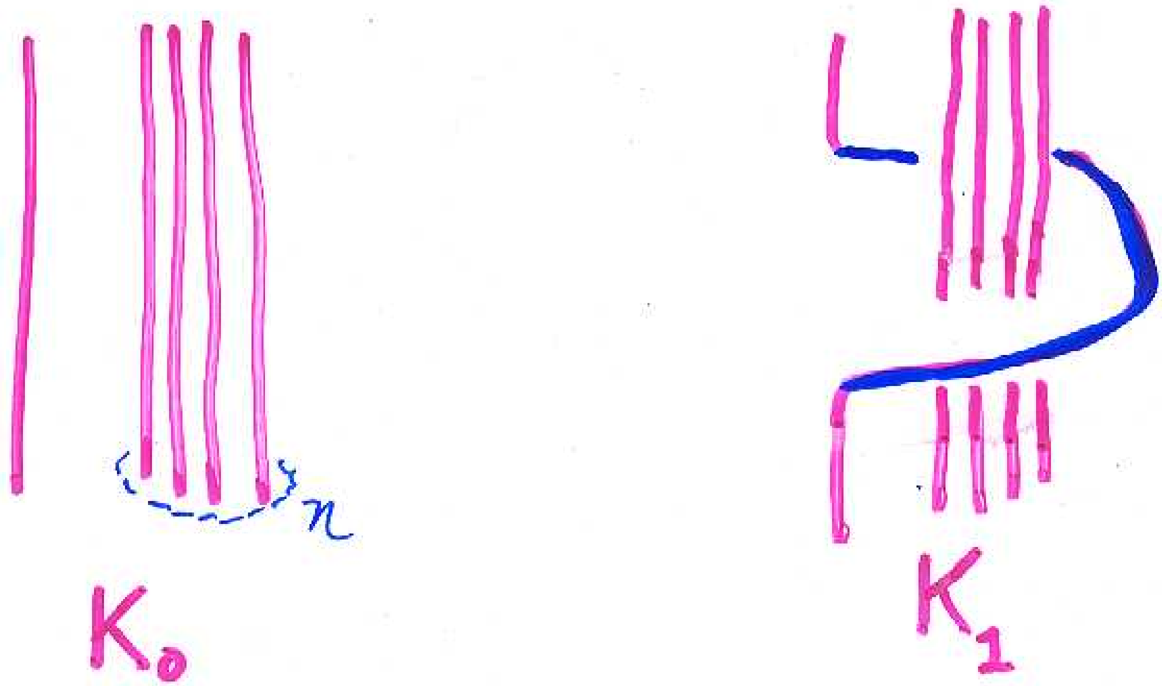


Any infection is a composition of <u>generalized crossing changes</u>

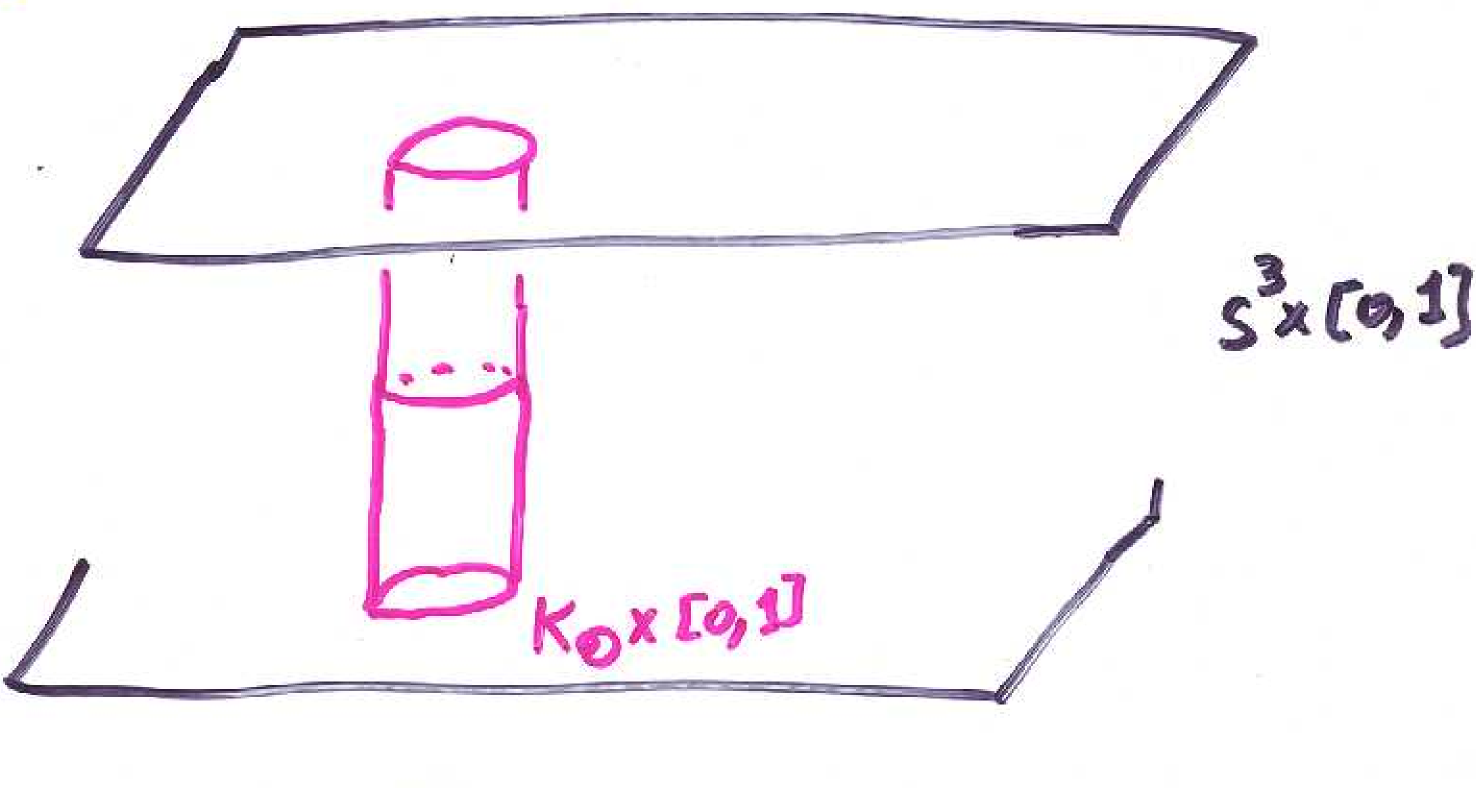


So it suffices to show that K_0, K_1 cobound n-grope in $S^3 \times [0,1]$ given η bounds n-grope in $S^3 \setminus K_0$

It suffices to show that the knots below cobound in $S^3 \times [0,1]$ an embedded grope of height n, assuming that



n bounds an embedded grope of height n in $S^3 \setminus K_0$.



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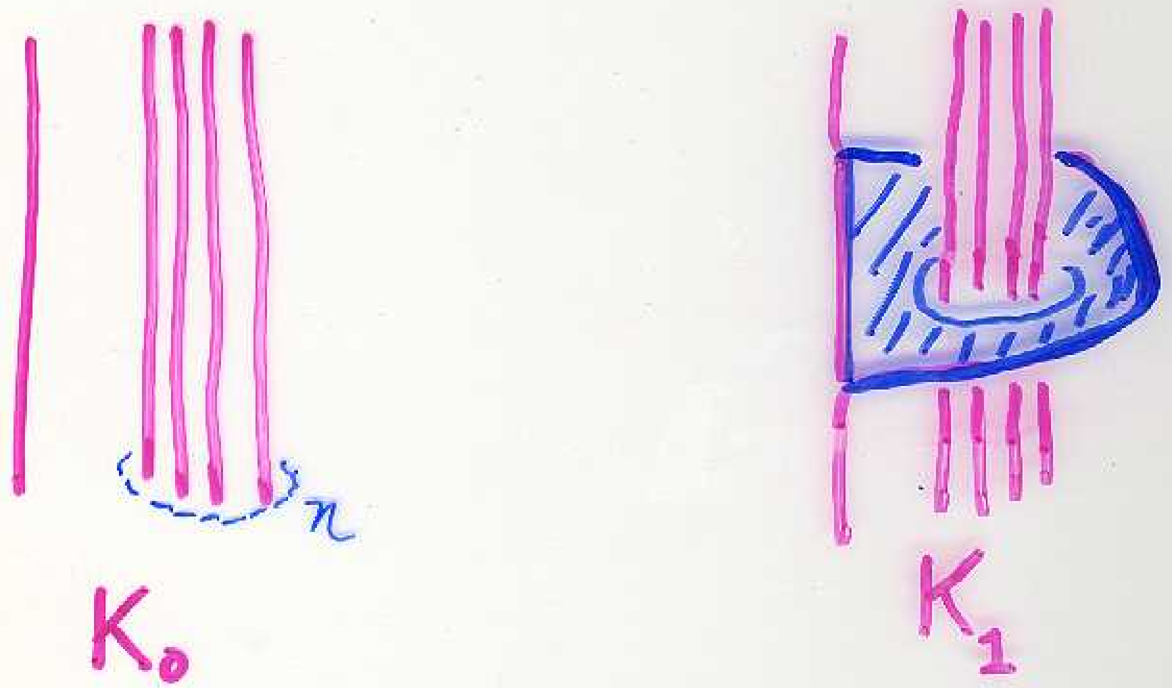
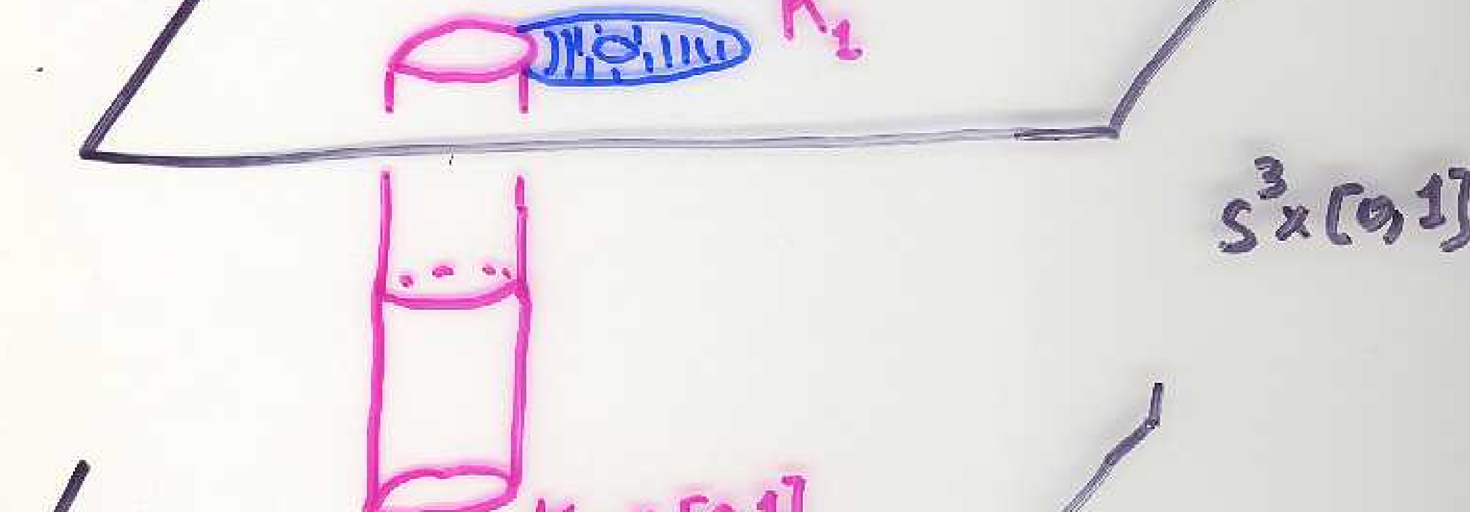


n bounds an embedded grope of height n in $S^3 \setminus K_0$.



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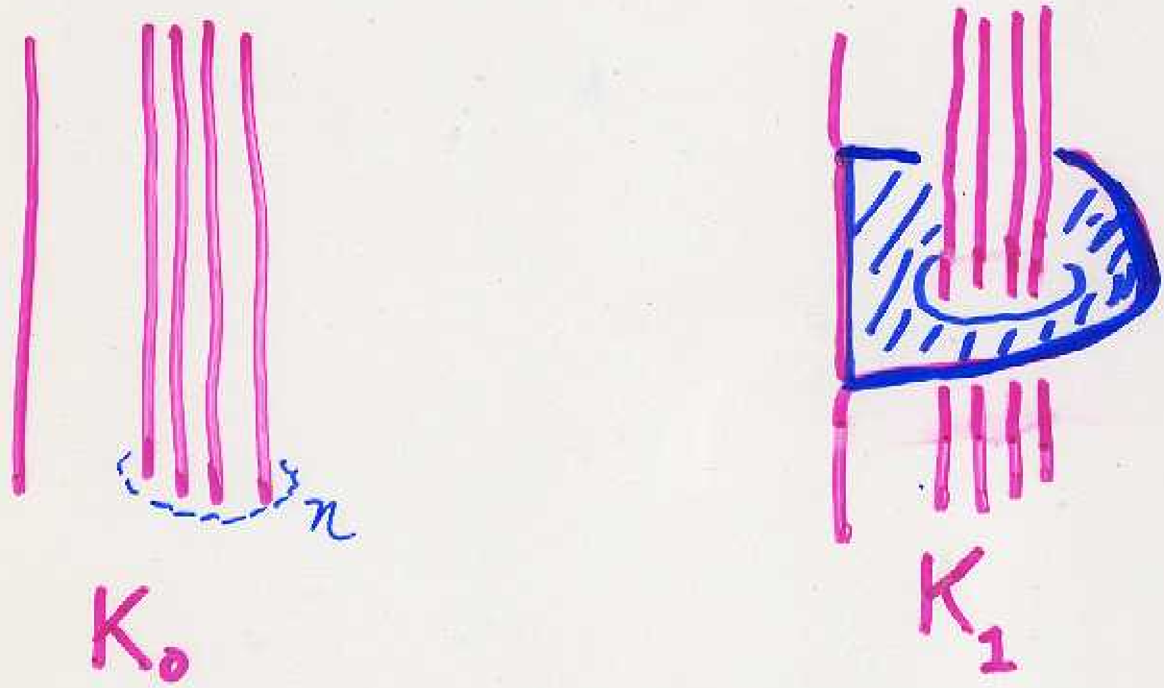
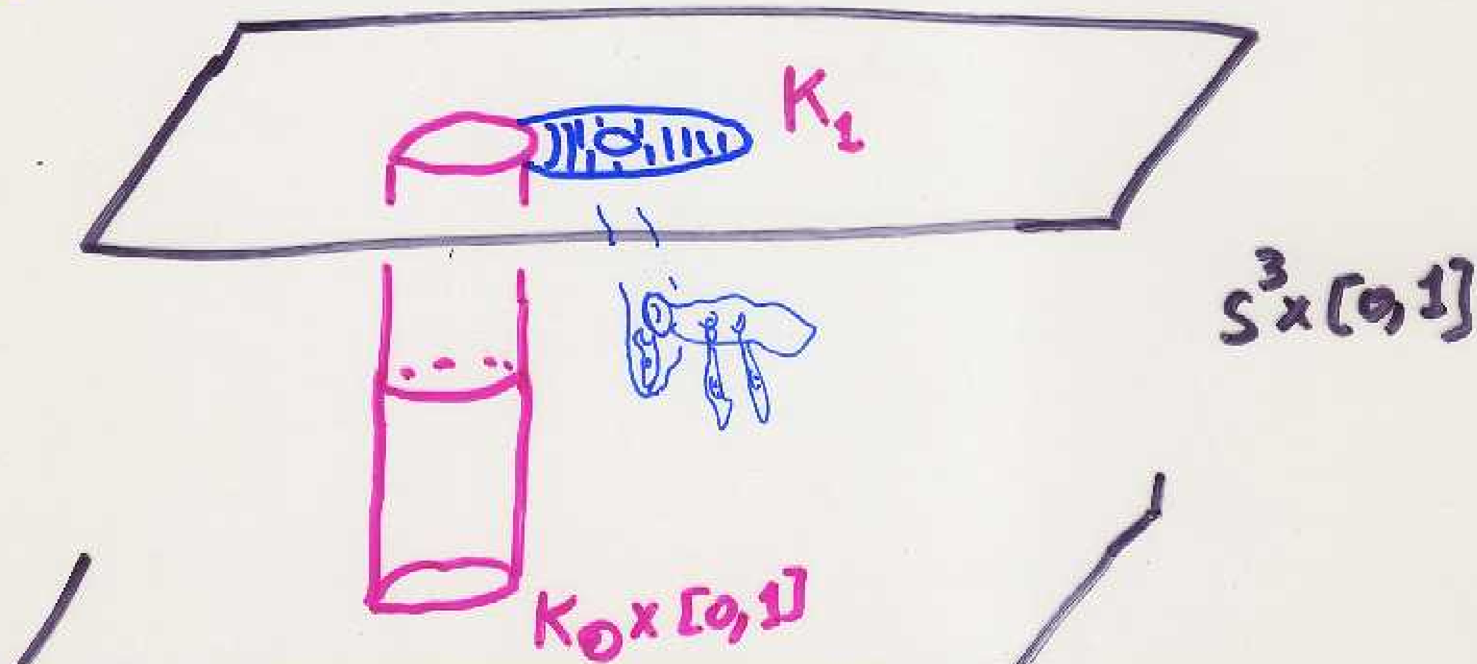


η bounds an embedded grope of height n in $S^3 \setminus K_0$.



OPEN PROBLEMS

1. TOP homology cobordism of $\mathbb{Z}$ homology lens spaces ??

2. $\bigcap_{n=1}^{\infty} \mathcal{G}_n = \{\text{slice knots and links}\}$

3. "Ribbon-Slice Problem" If a knot is concordant to a trivial Knot, is there a concordance $\pi_1(S^3 \setminus K) \twoheadrightarrow \pi_1(S^3 \times I - C$

4. Is there good notion of higher-order genus and Seifert form ?

5. torsion in $\mathcal{G}_n / \mathcal{G}_{n+1}$?

6. relate higher-order Alexander polynomials to some refinement of Osvath-Szabo-Rasmussen Floer homology.