Groping Around Link Concordance

Transparencies downloadable from my home page by Monday at latest.

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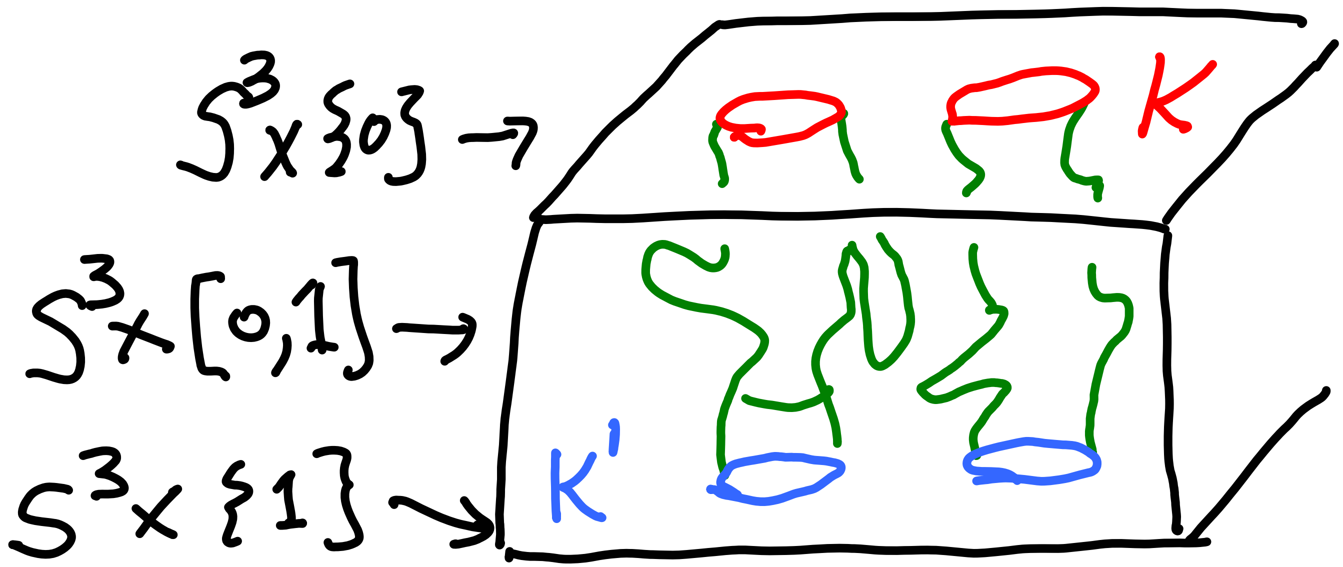
A link K will be an ordered, oriented collection of m circles in S^3 (tame).

Two links K, K' are concordant if they can be connected in $S^3 \times [0,1]$ by m oriented embedded annuli.



Smooth concordance = smoothly embedded.

Top concordance = TOP embedded with product neigborhood

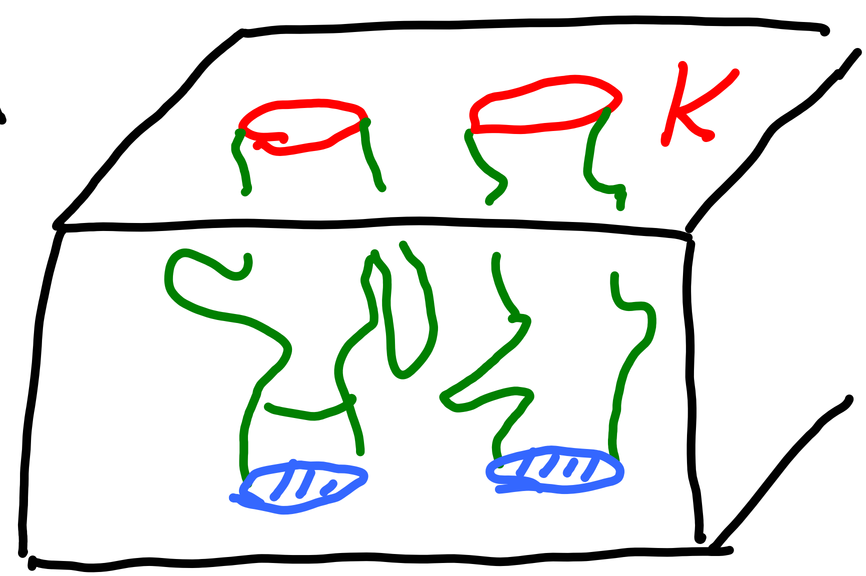
K is concordant to trivial link

$\Updownarrow$

K bounds embedded 2-disks in B^4

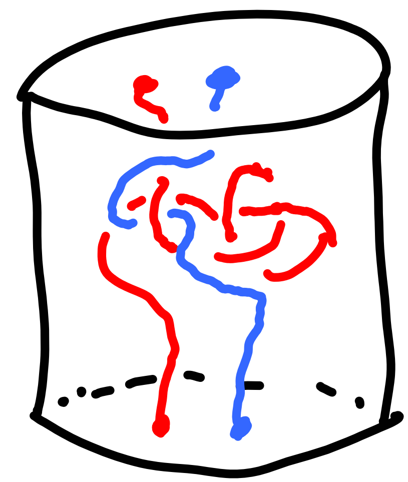
Such links are called

Slice Links



Links added by connected sum.
(for $m > 1$ must actually work with string links – suppressed in this talk

$D^2 \times [0,1]$



$-K$ = mirror image with orientations reversed

$K \# -K$ is always smoothly slice

$\mathcal{C}^m$ = group of concordance classes of m-component links under #

An abelian group if $m=1$ (knots)
non-abelian if $m>1$

ASIDE:
We have: $\mathcal{C}_{\text{SMOOTH}} \twoheadrightarrow \mathcal{C}_{\text{TOP}}$

<u>Thm</u> (Endo 1995) The kernel of this map contains a subgroup isomorphic to $\mathbb{Z}^\infty$.

specifically there is a family of pretzel knots all with Alexander polynomial 1 (hence TOP slice by Freedman) whose 2-fold branched covers are certain Brieskorn homology spheres shown by Furuta to be linearly independent in group of

$$\frac{\mathbb{Z}_2\text{-homology 3-spheres}}{\mathbb{Z}_2\text{-homology cobordism}}$$

using "gauge theory" (Fintushel-Stern).

Unknown: This kernel maps onto $\mathbb{Z}^\infty$ implying it has a $\mathbb{Z}^\infty$-summand.

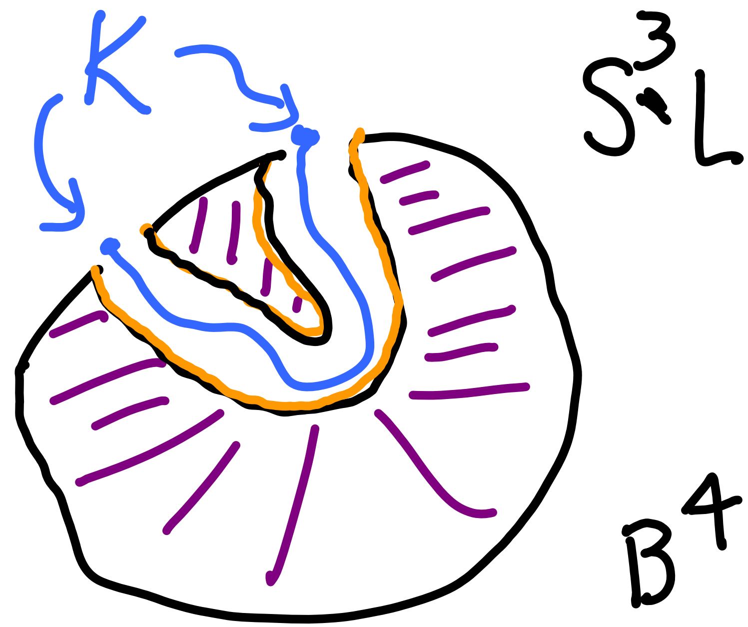
Thesis Project: Recover Endo's result using Khovanov-type homology or Osvath-Szabo-Rasmussen type techniques.

An approach to classical concordance invts

Prop. If K is a slice link then M_K, the 0-framed surgery on K, bounds a 4-manifold W such that

- $H_1(M_K) \xrightarrow{\cong} H_1(W) \cong \mathbb{Z}^m$
- $H_2(W) = 0$

Proof: Let W be $B^4 - N(\text{slice disks})$

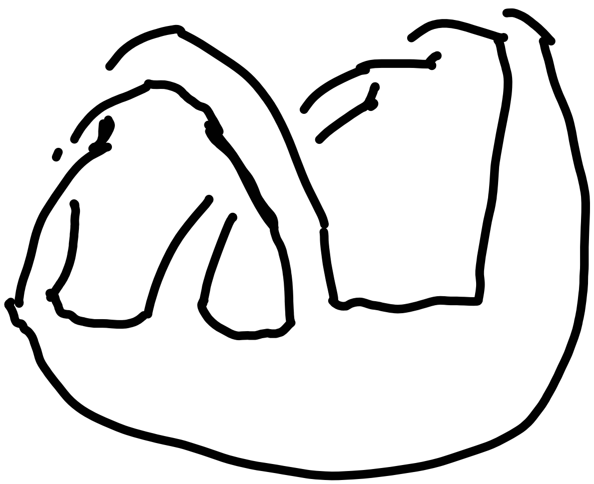


Exercise: almost if and only if

This observation leads to all classical invts:

1. $M_K = 0$ in $\Omega_3^{spin}(K(\mathbb{Z}^m, 1))$

- Arf invariants
- some Milnor invariants



$m=1$

FIGURE 8 KNOT

Arf Invariant $\neq 0$

$m=3$

Borromean Rings does not bound "over $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$".

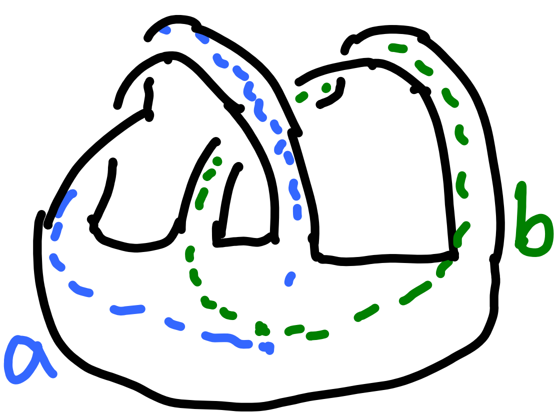
$\mathbb{Z}^m$-cover for general m

2. The ∞-cyclic cover of M_K is the boundary of the ∞-cyclic cover of W^4:

- signature invariants associated to the equivariant intersection form on $H_2(W_\infty)$ must vanish since $H_2(W) = 0$.

(Lemma$_{m=1}$: $\Rightarrow H_2(W_\infty)$ is Torsion module)

Levine-Tristram-Milnor signatures:



V = Seifert matrix of K
$= (lk(a_i, b_j^+))$

ω complex number of norm 1

ω-signature of K = signature $(\omega V + \bar{\omega} V^T)$

3. Equivariant Poincare DUALITY for (W_∞, M_∞) implies "$\frac{1}{2}$-lives and $\frac{1}{2}$ dies" in

$$H_1(M_\infty; \mathbb{Q}) \longrightarrow H_1(W_\infty; \mathbb{Q})$$

implying Alexander polynomial of a slice Knot factors

$$\Delta(t) \doteq f(t) f(t^{-1})$$

So: $\mathcal{C}^{m=1}_{TOP} \twoheadrightarrow \mathbb{Z}^\infty \times \mathbb{Z}_2^\infty \times \mathbb{Z}_4^\infty$

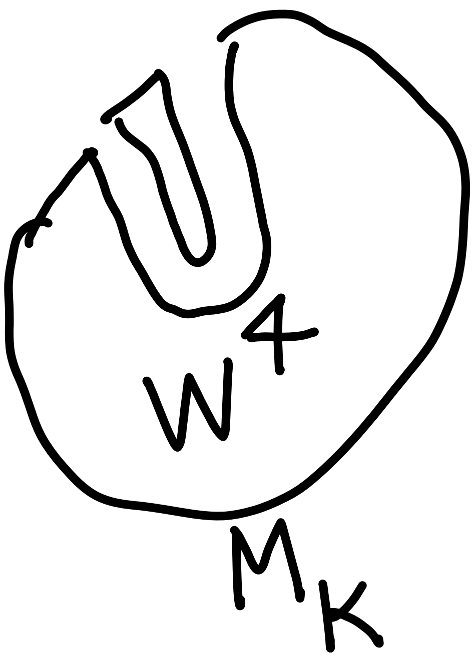
(Levine)

Kernel is called "algebraically slice knots"

These are all the classical knot concordance invariants.

How to get "higher-order" obstructions?

Find other groups Γ that can complete diagram:



$$\begin{tikzcd} \pi_1(M_K) \arrow[d, "i_*"] \arrow[r, twoheadrightarrow] & \Gamma \\ \pi_1(W) \arrow[ur, twoheadrightarrow] & \end{tikzcd}$$

Then we can get higher-order concordance invariants associated to Γ-covering space

- higher-order Arf invariants (bordism)
- " " signatures (L^2-signatures)
- " " Alexander Polynomials
- " " Alex. Modules ($H_1(\Gamma\text{-cover})$)

1973 Γ = metabelian Casson-Gordon defined signatures associated to Γ

$\Rightarrow$ {Algebraically Slice} $\supseteq \mathbb{Z}^\infty \times \mathbb{Z}_2^\infty$

(Casson-Gordon, Jiang, Livingston)

'99 program works for Γ = torsion-free-solvable group (C-Orr-Teichner)

~2002 $\Rightarrow$ $\mathbb{Z}^\infty \subseteq \left\{ \begin{array}{l} \text{KNOTS with vanishing} \\ \text{CASSON-Gordon invts} \end{array} \right\}$

$$\Gamma = \pi_1(W) / \pi_1(W)^{(3)}$$

Aside: Higher-order Alexander modules used by Harvey, Cochran are $\dfrac{\pi^{(n)}}{\pi^{(n+1)}}$ associated to iterated universal abelian covers of M_K

These led to new invariants δ_n that are degrees of higher-order Alexander polynomials and give sharper estimates than usual Alexander polynomial for genus(K) or Thurston norm.

$$\delta_0 \leq \delta_1 + 1 \leq \delta_2 + 1 \leq \ldots \leq 2\,\mathrm{genus}(K)$$

There should be a refinement of Link FLOER Homology where these δ_n can be "read off", but none of these degrees of Alex. polynomials is a concordance invt. So they will not appear today.

ANOTHER CLASSICAL SET OF CONCORDANCE INVTS:

A different crucial advance was taken in 1963 by John Stallings but applies only to links $m > 1$.

RECALL: K slice $\Longrightarrow \exists\, W$ such that $S^3 \setminus K \to W = B^4 - \Delta$ is

__Thm__ (Stallings) If $f: M \longrightarrow W$ is a homology isomorphism.

- isomorphism on $H_1(-;\mathbb{Z})$
- epimorphism on $H_2(-;\mathbb{Z})$

then $\forall\, n$ $\dfrac{\pi_1(M)}{\pi_1(M)_n} \cong \dfrac{\pi_1(W)}{\pi_1(W)_n}$

A_n = n^{th} term of lower central series of group A

$A_1 = A$, $A_{n+1} \equiv [A, A_n]$.

Consequently: If K is a slice link then

$$\frac{\pi_1(S^3 \setminus K)}{\pi_1(S^3 \setminus K)_n} \cong \frac{F\langle x_1, .., x_m \rangle}{F_n} \quad \forall\, n$$

and longitudes $\in \pi_1(S^3 \setminus K)_n \ \forall\, n$,

so Milnor's $\bar{\mu}$-invariants are concordance invariants.

Also can now use $\Gamma = \pi/\pi_n$ to get signature obstructions and ranks of Alexander modules associated to this Γ

$$\begin{tikzcd} \pi_1(M_K) \arrow[r, two heads] \arrow[d] & \pi/\pi_n \\ \pi_1(W) \arrow[ur] & \end{tikzcd}$$

(Smolinsky)

THIS CONCLUDES MY INTRO TO LINK CONCORDANCE INVARIANTS

Other Equivalence Relations

- give more information about structure in $\mathcal{C}_{TOP}$ and $\mathcal{C}_{smooth}$

Idea: commutator series filter a group G

$$\cdots \subseteq G_3 \subseteq G_2 \subseteq G$$

and to each commutator series there is a family of equivalence relations filtering the link concordance group.

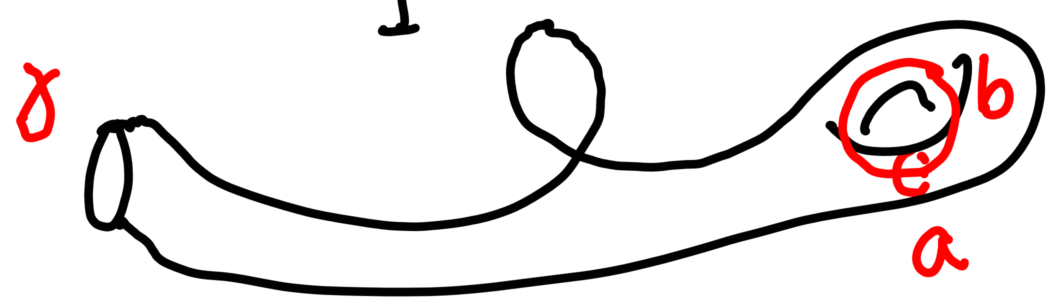
Derived Series of G: $G^{(0)} \equiv G \quad G^{(1)} = G' = [G,G]$

$$G^{(n+1)} \equiv [G^{(n)}, G^{(n)}]$$

there are other commutator series.

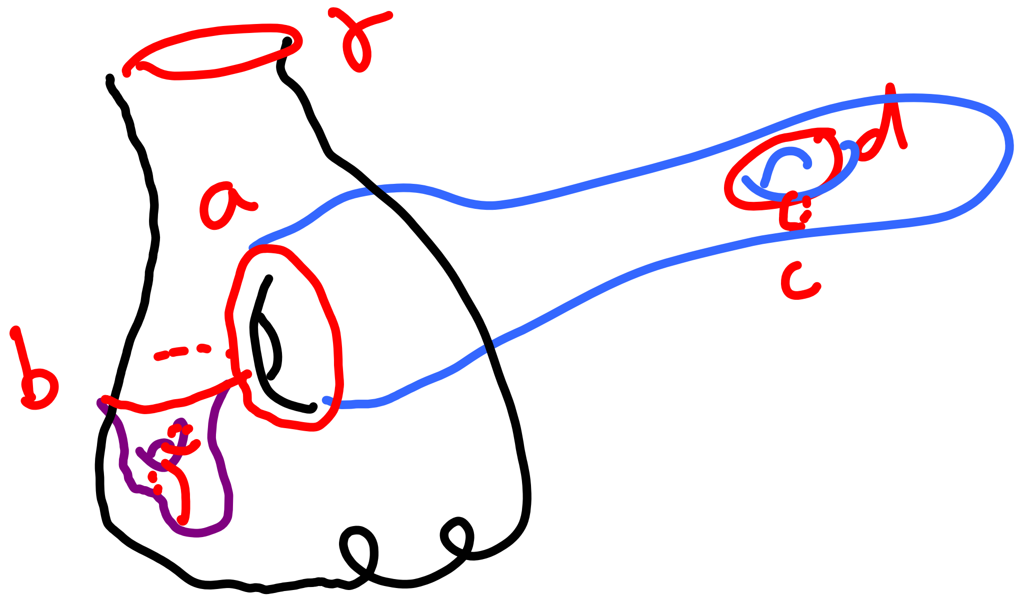
<u>Basic Idea</u>: a circle γ is a commutator

$\gamma = [a,b]$ in $\pi_1(X) \iff \gamma$ bounds a surface (NOT embedded)



Similarly $\gamma = [[c,d],[e,f]]$ in $\pi_1(X)$

$\iff \gamma$ bounds a surface



$\gamma = [[c,d],[e,f]]$

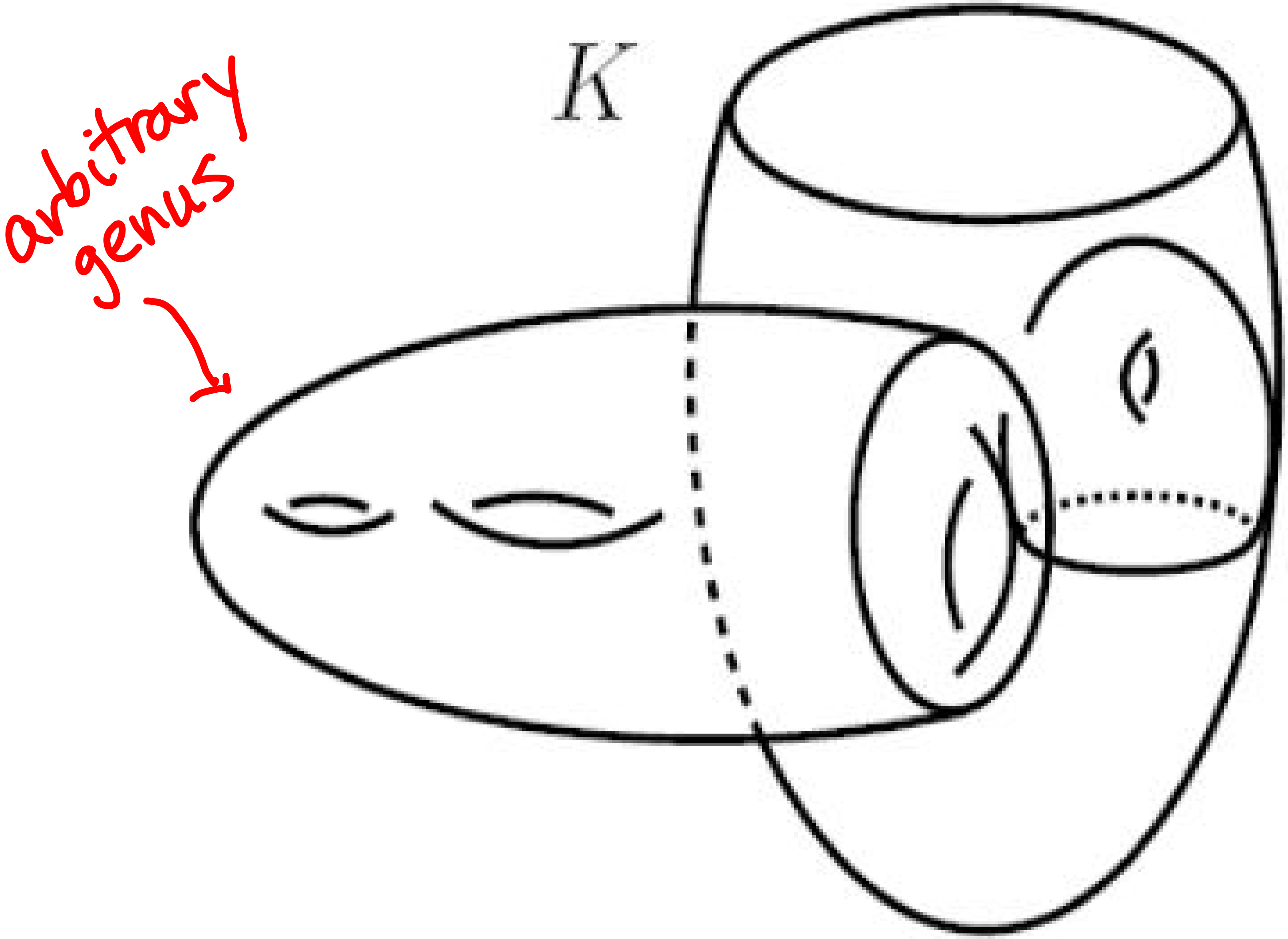
We call such 2-complexes consisting of layers of surfaces glued to elements of a symplectic basis of a previous surfaces:

Gropes (different type for each series)

<u>Symmetric or full-grope</u> corresponds to derived series: attach surface to all elements of symplectic basis of previous surfaces.

<u>HALF-Grope</u>: corresponds to lower central series: attach surfaces to 1/2 of symplectic basis.

Which links bound <u>embedded</u> gropes in S^3 or B^4?



Symmetric Grope height 2

- Knots that <u>co</u>bound, with the unknot, half-gropes in S^3 with n-stages are those knots with vanishing Vassiliev invts. of order $\leq n$ (± 1) (Conant-Teichner, Habiro)
- links that bound ½-gropes in B^4 are intimately related to Milnor's $\bar{\mu}$-invariants (Cochran, Krushkal-Teichner)

We focus on symmetric gropes related to derived series

Let $\mathcal{G}_n^m$ = concordance classes of m-comp. links that bound embedded height n symmetric gropes in B^4

$$\{e\} \subseteq \cdots\cdots \subseteq \mathcal{G}_2^m \subseteq \mathcal{G}_1^m \subseteq \mathcal{C}^m$$

$\{e\}$ = SLICE LINKS

__Previous RESULTS on KNOTS__ $m=1$ (C-Orr-Teichner)

$K \in \mathcal{G}_3 \Rightarrow K$ is algebraically slice

$K \in \mathcal{G}_4 \Rightarrow K$ has vanishing Casson-Gordon

$\Rightarrow \mathbb{Z}^\infty \subseteq \mathcal{G}_4 / \mathcal{G}_5$

- $\mathcal{G}_n / \mathcal{G}_{n+1}$ is infinite $\forall\, n \geq 2$ (C-Teichner)
- For any knot K with degree $\Delta_K(t) > 2$, and any $n \geq 2$ there is an infinite set of knots each smoothly n-grope cobordant to K but all distinct up to $(n+1)$-grope cobordism and all with the same s and τ invariants (O-S Rasmussen).
(C-T.Kim)

(examples have <u>same</u> genus so all $s, \tau \in \{-2g, \ldots, 0, \ldots, 2g\}$)

RESULTS FOR LINKS: $m \geq 2, n \geq 1$

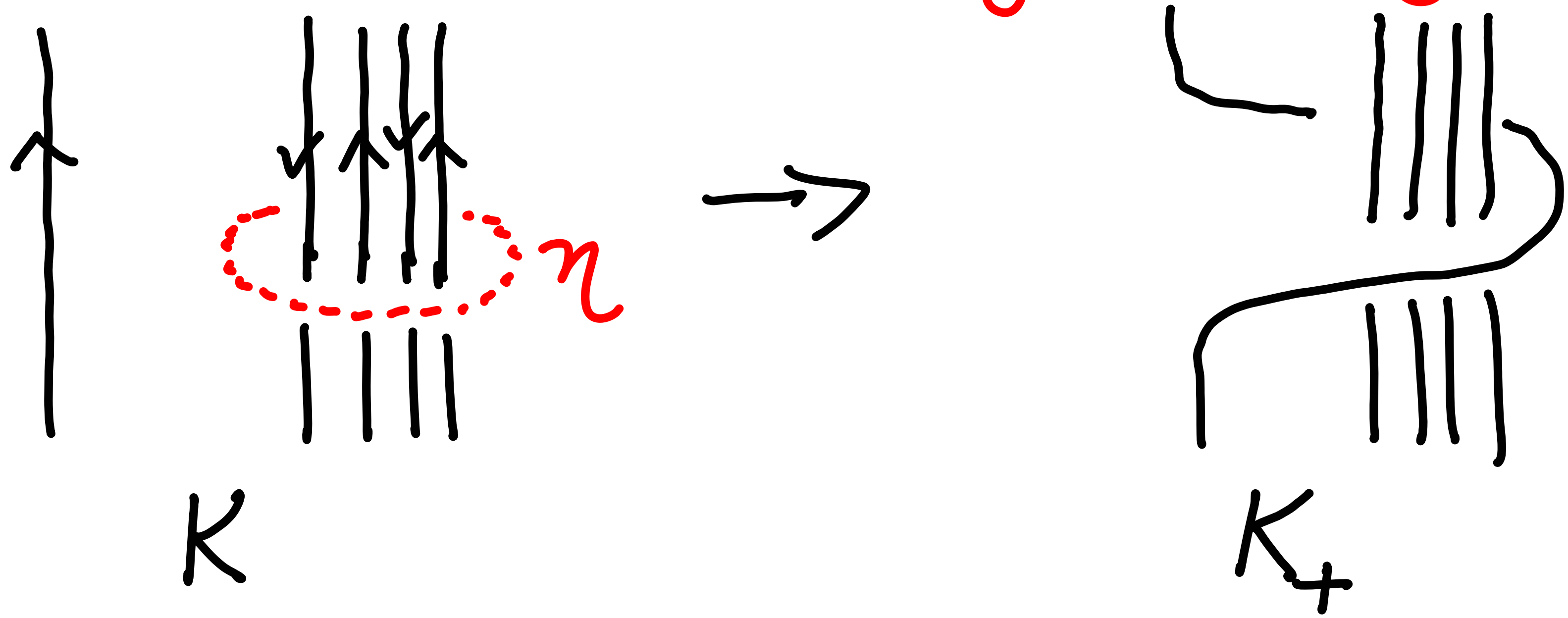
- $\mathcal{G}_n / \mathcal{G}_{n+2}$ contains a subgroup consisting entirely of boundary links that maps onto $\mathbb{Z}^{\infty}$. (Harvey)
- Milnor's invariants give maps

$$\mathcal{G}_n / \mathcal{G}_{n+1} \twoheadrightarrow \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{\text{some finite \#}}$$

How to create knots and links that bound height n symmetric gropes in B^4

(C-Teichner)

Generalized Crossing Change

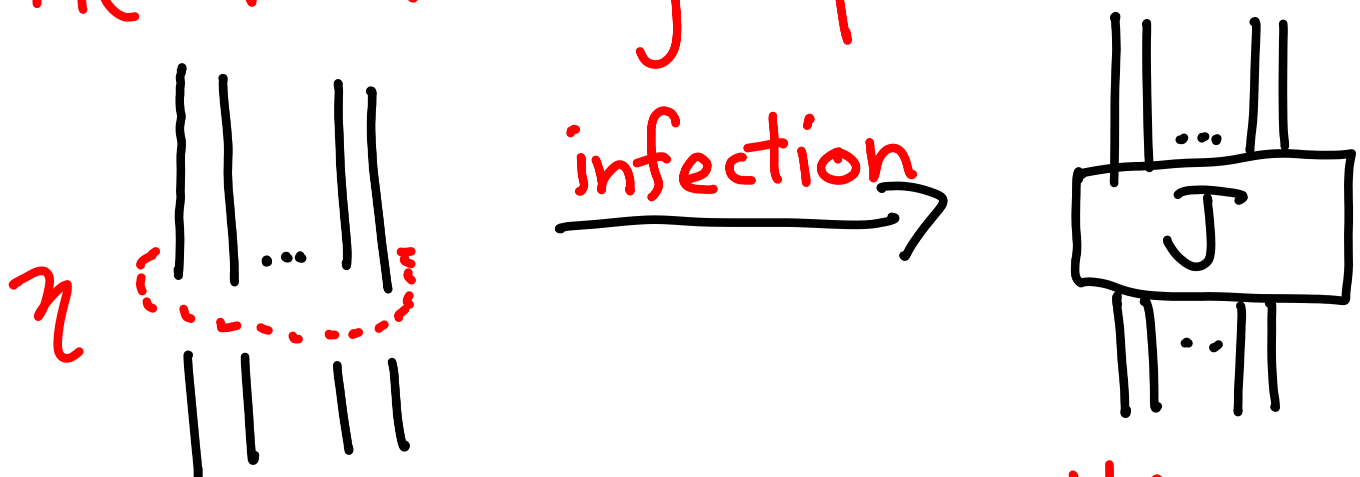


* Suppose η bounds embedded n-grope in $S^3 \setminus K$. (in particular $\eta \in \pi^{(n)}$)

2^{n+1}, clasper

Prop [C-T]: K and K' are n-grope cobordant so if K was trivial or a slice link then K' bounds embedded n-grope in B^4!

Corollary: The same is true for the following operation

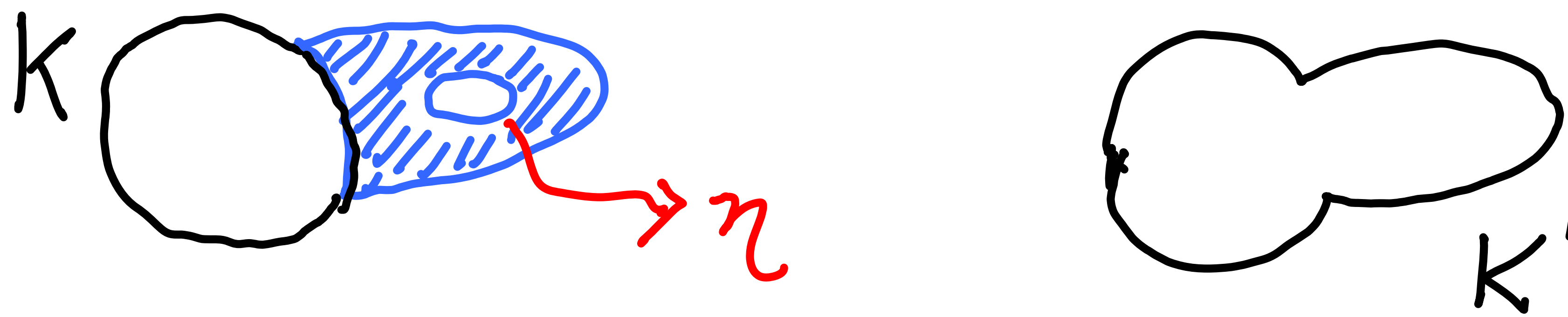


since it is a composition of generalized crossing changes!

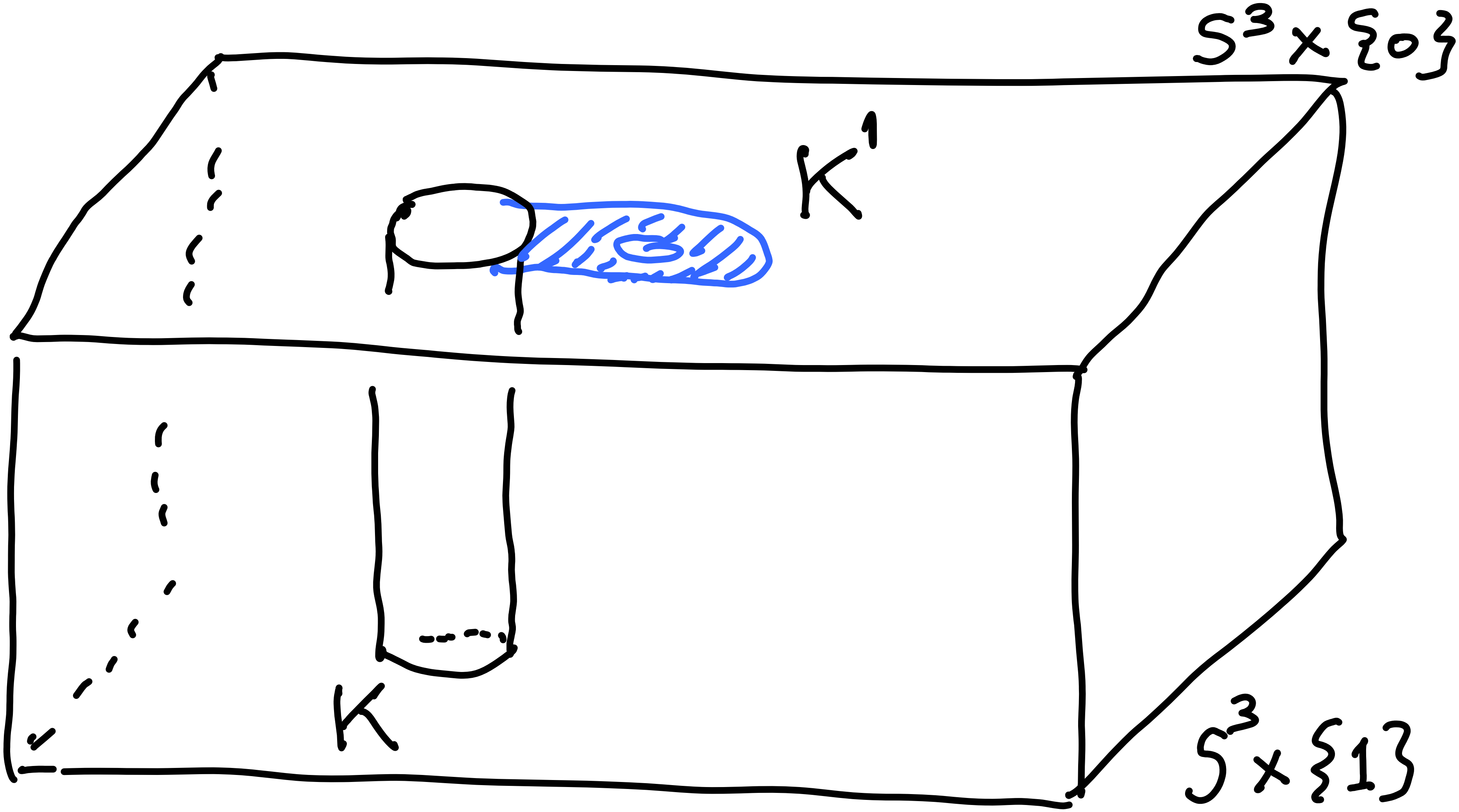
Proof that K and K' are n-grope cob.

1. Start with product concordance
$$K \times [0,1] \subseteq S^3 \times [0,1]$$

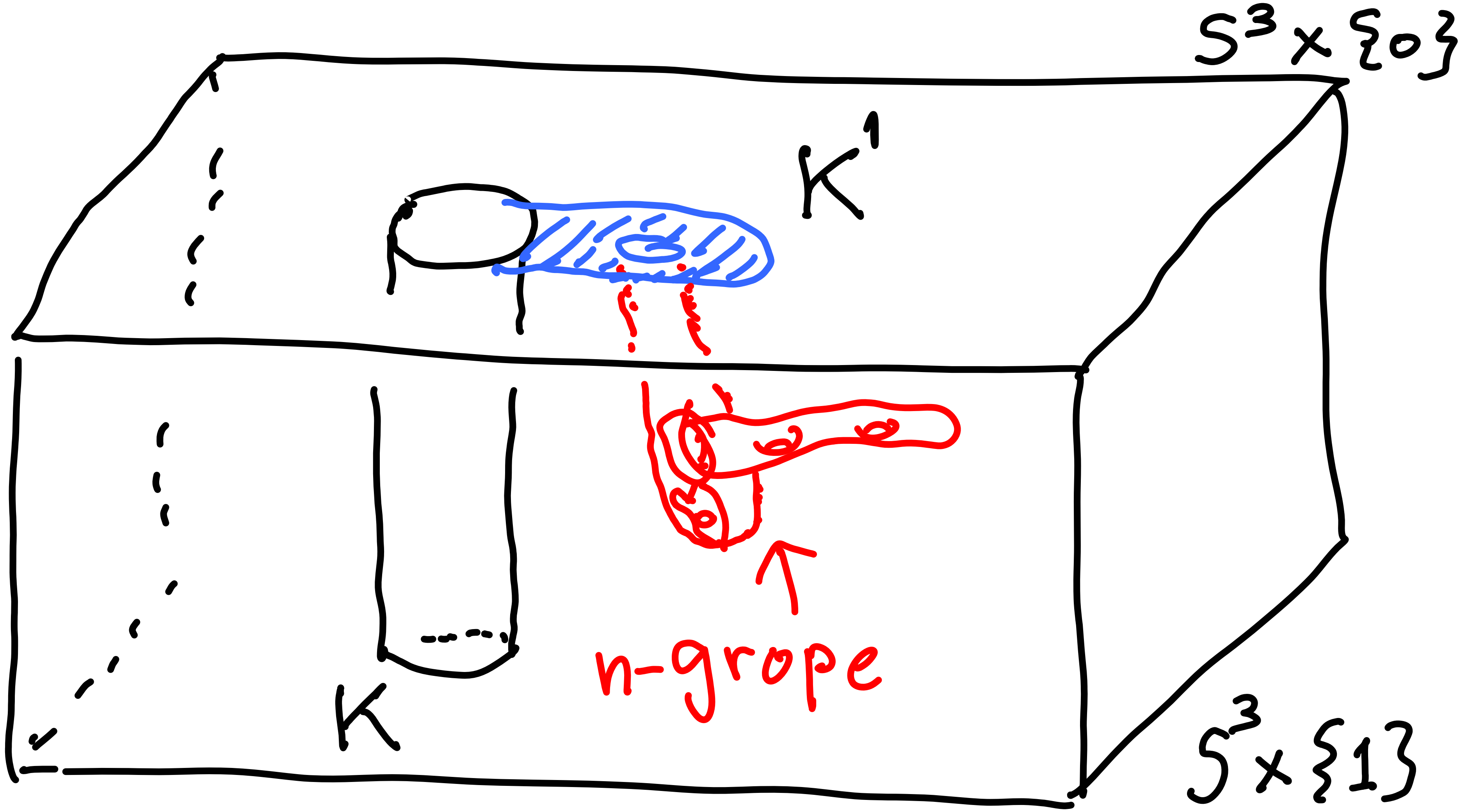
2. Add Blue punctured disk to $K \times \{0\}$



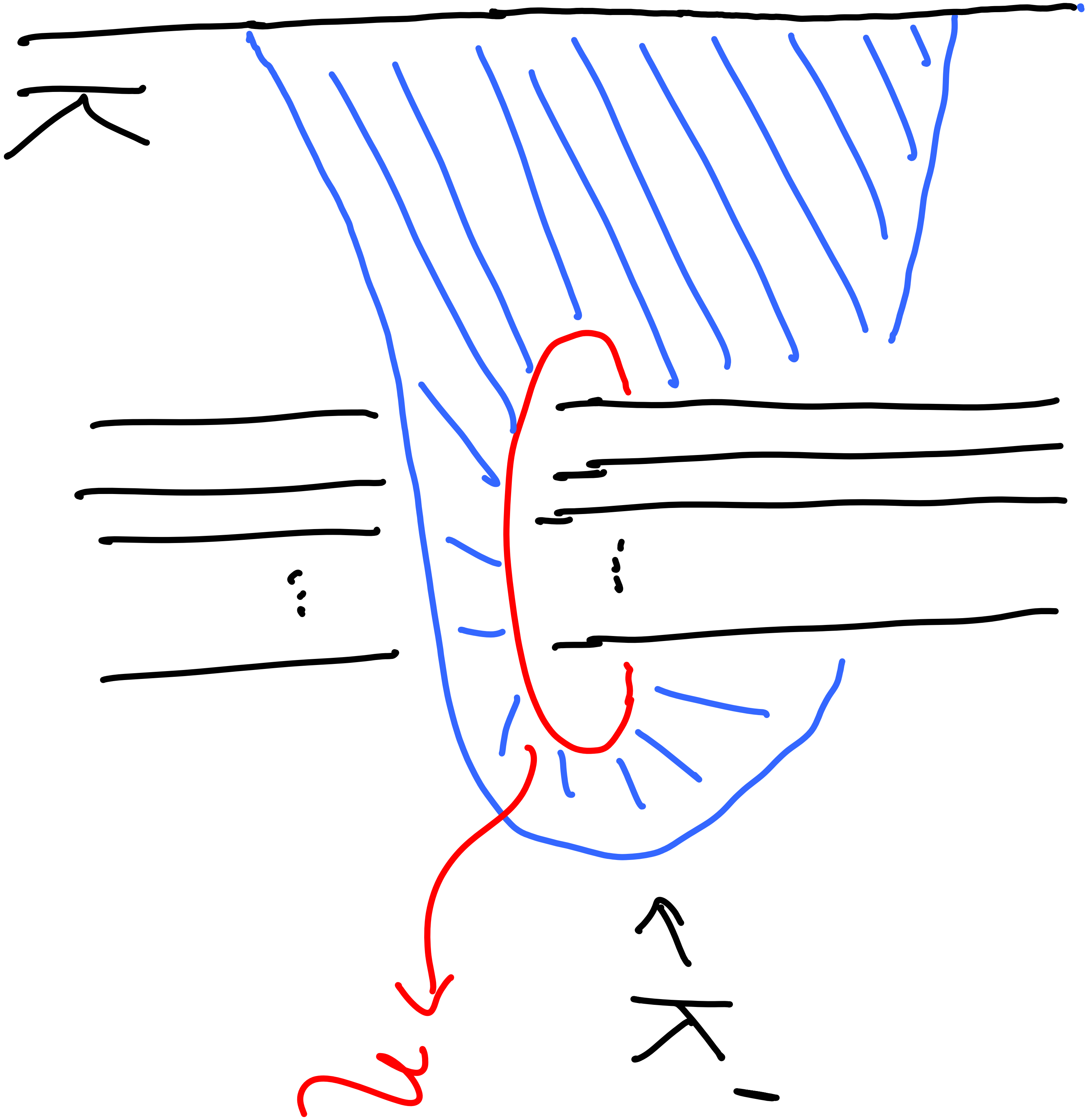
such that puncture is η
and "new boundary" is K'



$S^3 \times \{0\}$
K'
K
$S^3 \times \{1\}$

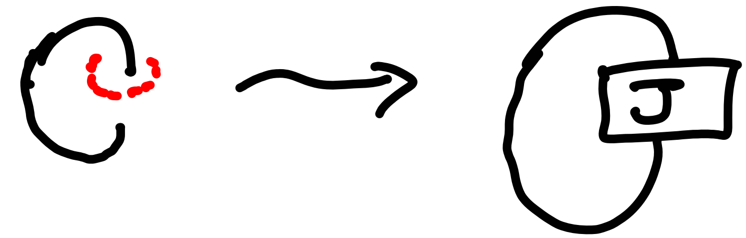


Result is n-grope cobordism from K' to K

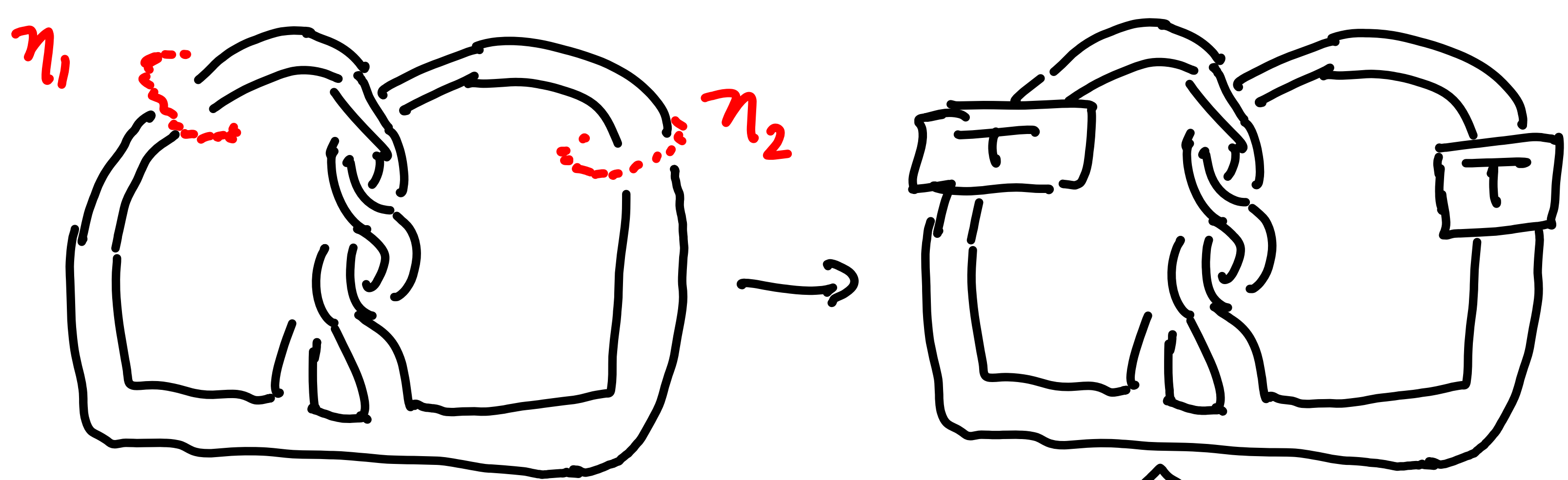


Examples:

$n=0$



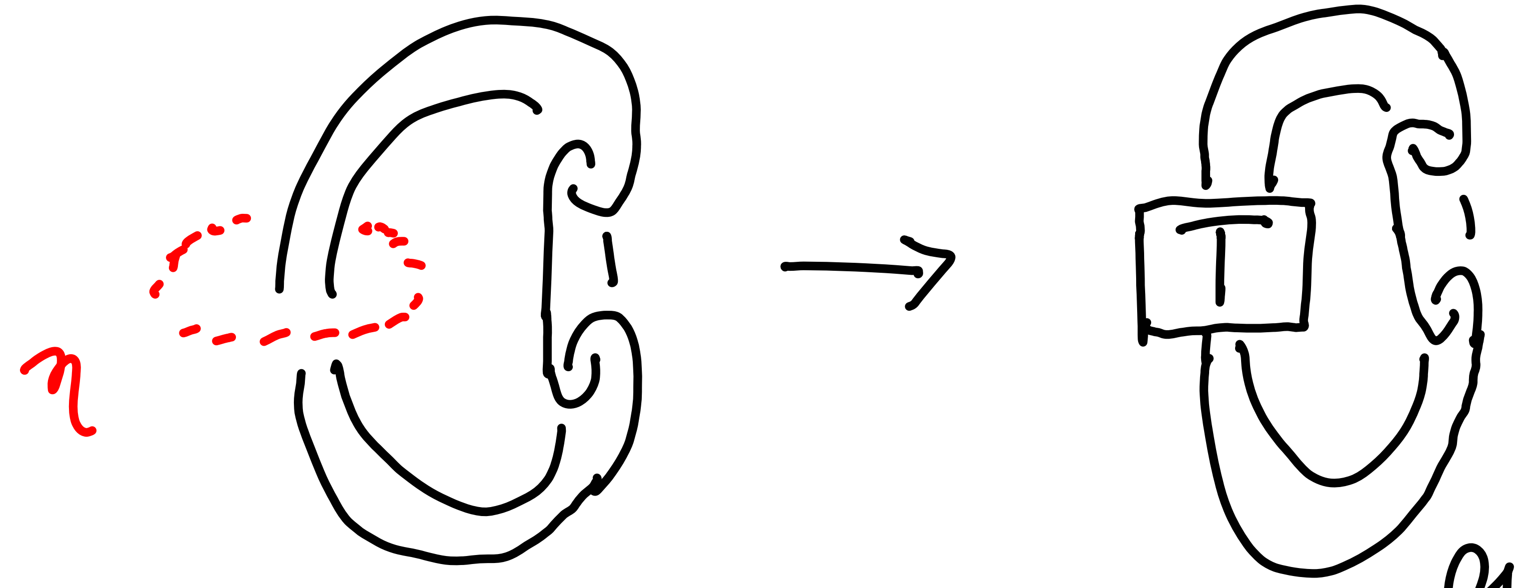
$n=1$



doesnt bound

a 4-grope by higher-order signatures

$n=1$



Bing Double of Trefoil

not in $\mathcal{G}_4$ by higher-order signature invariant (Harvey)

For arbitrary $n \geq 0$ it is always possible to find such η for trivial link $(m>1)$

$\eta \in F^{(n)}$

$\eta \notin F^{(n+1)}$

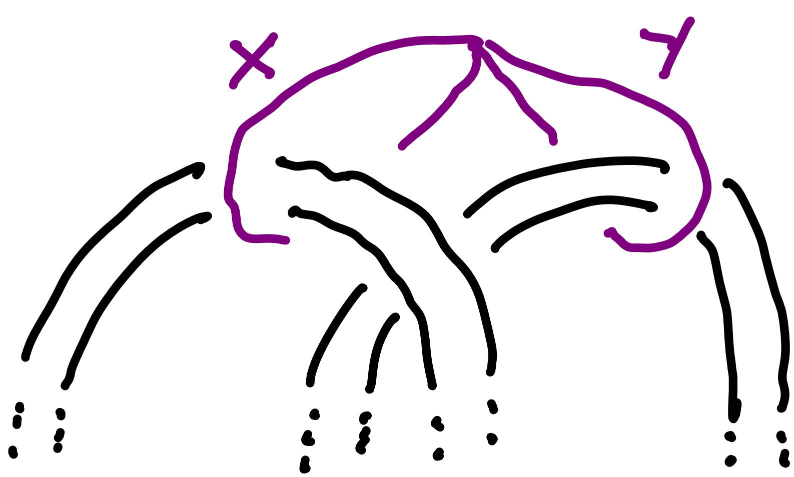
since $\pi_1 =$ Free group and $\bigcap_{n=0}^{\infty} F^{(n)} = \{e\}$

Infecting by trefoil produces link that bounds an n-grope by Proposition but doesn't bound $(n+3)$-grope (Harvey) in particular is not slice.

For any <u>slice</u> Knot K, $\Delta_K(t) \neq 1$,

the same can be done

Choose x, y independent in Alex. module of K:



$=$

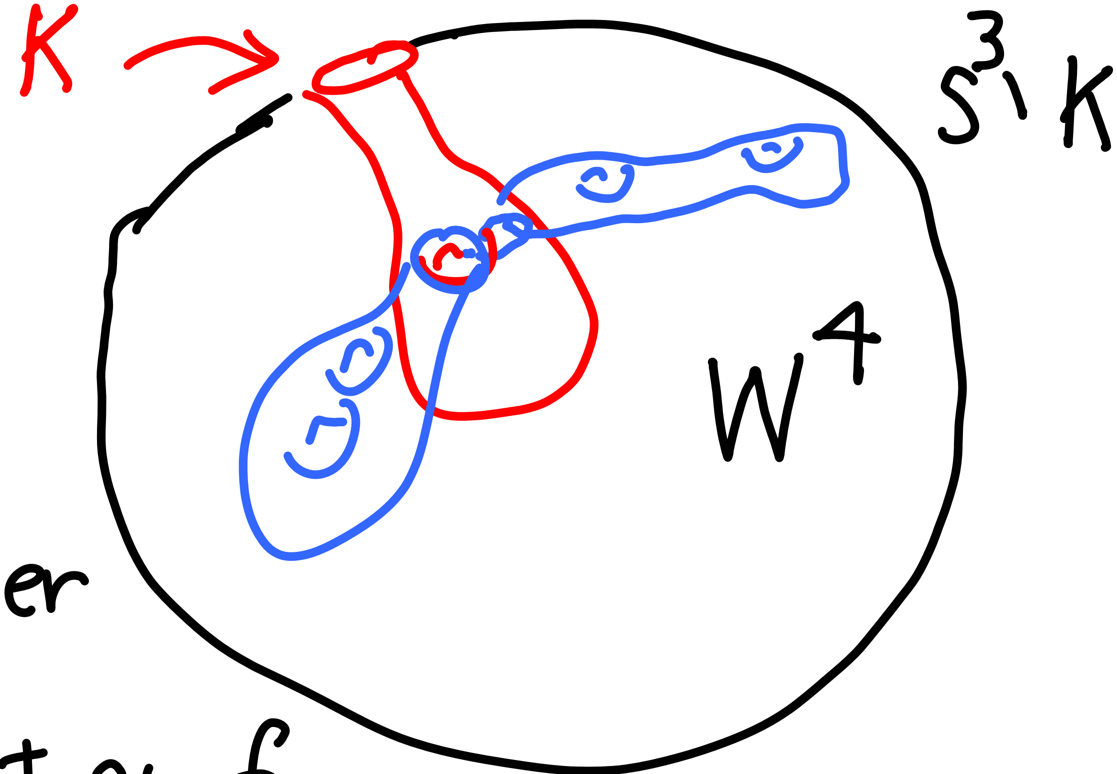
choose $\eta \in F^{(n)} - F^{(n+1)}$

Prop. (C-): $\eta \in G^{(n+1)} - G^{(n+2)}$ $\quad G = \pi_1(S^3 \setminus K)$

This is the way one produces links that are non-trivial at each stage of the Grope filtration $\begin{pmatrix} m=1 & \text{C-Teichner} \\ m>1 & \text{Harvey} \end{pmatrix}$

But why are there invariants that obstruct bounding long gropes?

Basic Idea:



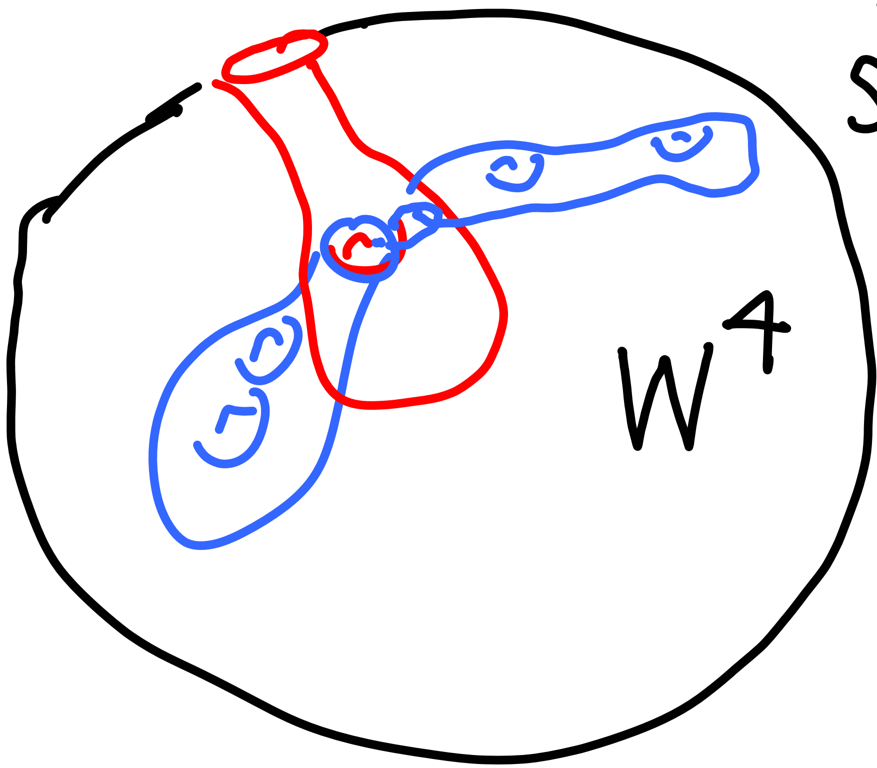
Suppose K bounds an (n+2)-grope in B^4. Consider

$W^4 = B^4$ - first surfaces

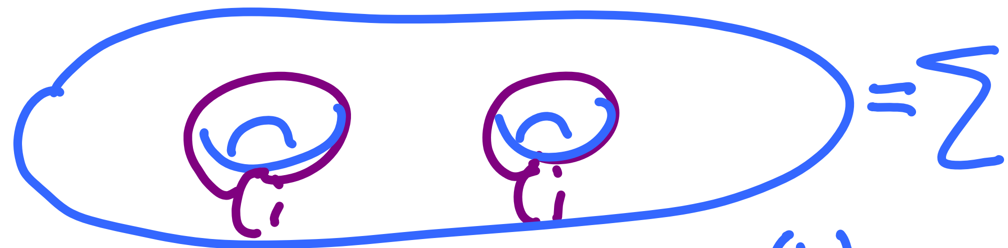
then $H_1(S^3 \setminus K) \cong H_1(W) \cong \mathbb{Z}^m$

as before but now $H_2(W) \neq 0$,

so signatures maybe $\neq 0$, No Stallings Thm.



Key: $H_2(W^4)$ is generated by n-surfaces



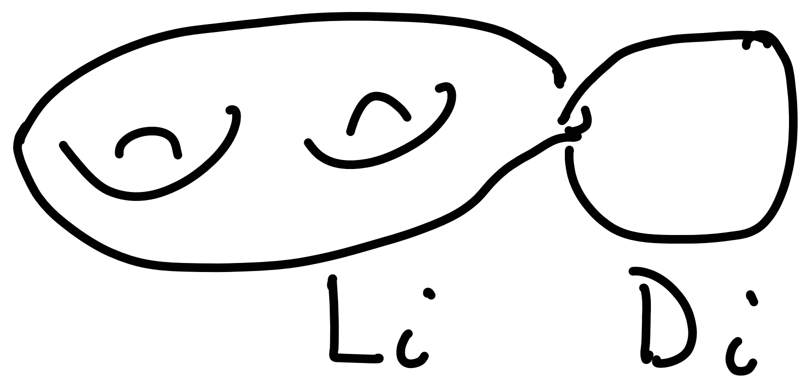
$\pi_1(\Sigma) \subseteq \pi_1(W)^{(n)}$

Is there an analog of Stalling's theorem for derived series <u>and</u> with H_2 hypothesis weakened?

with more work one can show if K bounds $(n+2)$-grope then $M_K = \partial W^4$ such that

- $H_1(M) \cong H_1(W)$
- $H_2(W)$ is generated by embedded n-surfaces L_i, D_i that intersect

so $\bigoplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



Thm (Stallings) If $f: M \longrightarrow W$ is

- isomorphism on $H_1(-;\mathbb{Z})$
- epimorphism on $H_2(-;\mathbb{Z})$

then $\forall\, n$ $$\frac{\pi_1(M)}{\pi_1(M)_n} \cong \frac{\pi_1(W)}{\pi_1(W)_n}$$

There IS an analogue of Stallings' theorem for derived series and the hypothesis on H_2 can be weakened to what we need!

$G_H^{(n)}$ = torsion-free derived series (Harvey)

<u>Theorem</u> (C-Harvey) $f: M \to W$

- monomorphism $H_1(M;\mathbb{Q}) \to H_1(W;\mathbb{Q})$
- epimorphism $H_2(M;\mathbb{Q}) \twoheadrightarrow \underbrace{H_2(W;\mathbb{Q})}_{\langle n\text{-surfaces}\rangle}$

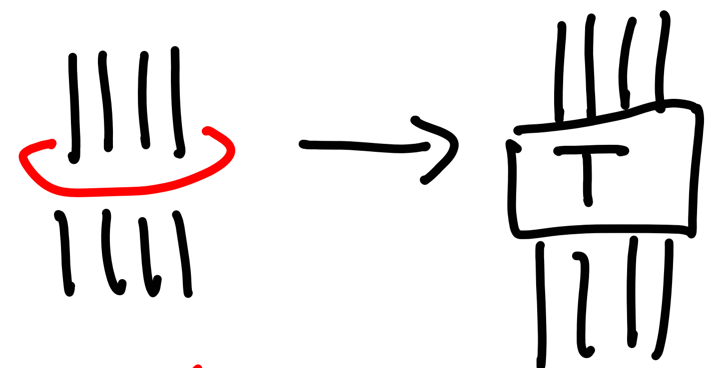
then $$\frac{\pi_1(M)}{\pi_1(M)_H^{(n+1)}} \hookrightarrow \frac{\pi_1(W)}{\pi_1(W)_H^{(n+1)}}.$$

(Dwyer)

Why are such injectivity results crucial? Recall we chose

$\eta \in \dfrac{\pi(S^3 \setminus K)^{(n)}}{\pi(S^3 \setminus K)^{(n+1)}}$ non-zero and

did "infection"



but REALLY NEED $i_*(\eta) \neq 0$

$$\frac{\pi_1(S^3 \setminus K)^{(n)}}{\pi_1(S^3 \setminus K)^{(n+1)}} \xrightarrow{i_*} \frac{\pi_1(W)^{(n)}}{\pi_1(W)^{(n+1)}}$$

Applications:

- show ranks of certain higher-order Alexander modules are invariants of n-grope cobordism
- improve on Harvey's results on grope filtration of link concordance
- show some results of COT work for non-manifolds