

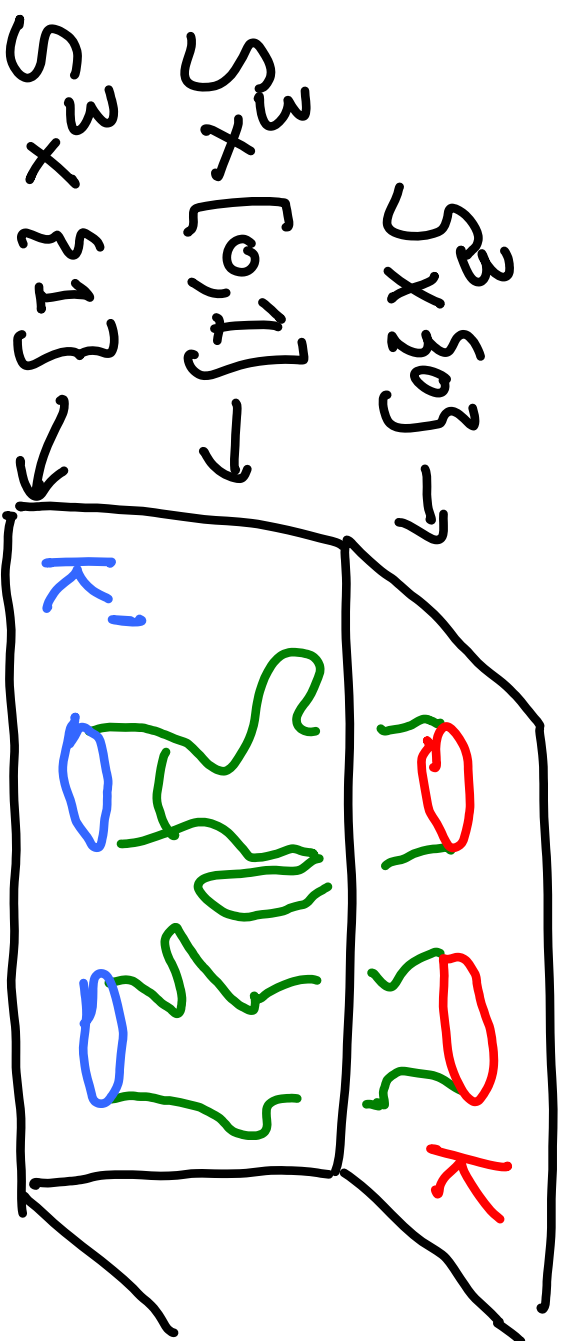
Groping Around Link Concordance

Transparencies downloadable from my
home page by Monday at latest!

Tim Cochran
Rice University

A link K will be an ordered, oriented collection of m circles in S^3 (frame).

Two links K, K' are concordant if they can be connected in $S^3 \times [0,1]$ by m oriented embedded annuli.



Smooth concordance = Smoothly embedded.

Top concordance = Top embedded with product neighborhood

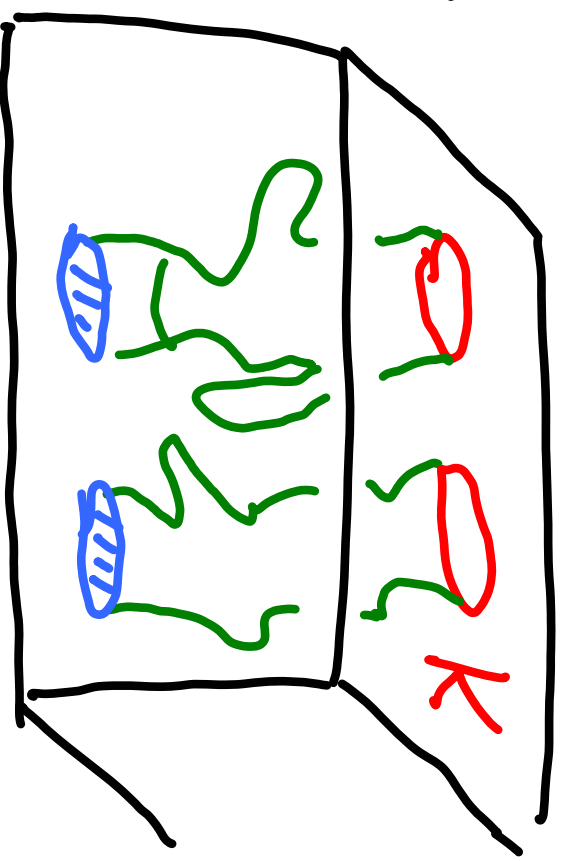
K is concordant to trivial link



K bounds embedded 2-disks in B^4

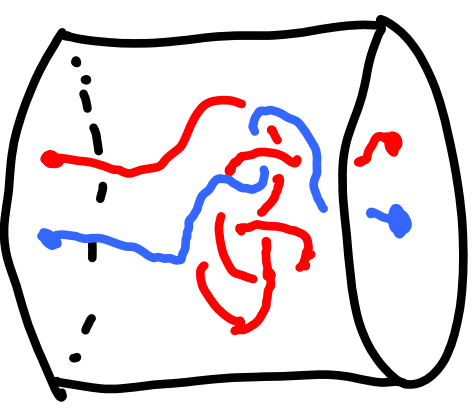
Such links are called

Slice Links



Links added by connected sum.
(for $m > 1$ must actually work with
string links - suppressed in this talk

$D^2 \times [0,1]$



- K = mirror image with orientations
reversed
 $K \# -K$ is always smoothly slice

$\mathcal{C}_m =$ group of concordance classes of m -component links under $\#$
An abelian group if $m = 1$ (knots)
non-abelian if $m > 1$

*SIDE:

We have: $\mathcal{C}_0 \xrightarrow{\text{SMOOTH}} \mathcal{C}_0^{\text{TOP}}$

Thm (Endo 1995) The kernel of this map contains a subgroup isomorphic to \mathbb{Z}_2^∞ .

specifically there is a family of pretzel
knots all with Alexander polynomial 1
(hence TOP slice by Freedman) whose
2-fold branched covers are certain
Brieskorn homology spheres shown
by Furuta to be linearly independent
in group of \mathbb{Z}_2 -homology 3-spheres
 \mathbb{Z}_2 -homology cobordism
using "gauge theory" (Fintushel-Sten).

Unknown: This kernel maps onto \mathbb{Z}_1^∞ implying it has a \mathbb{Z}_1^∞ -summand.

Thesis Project: Recover Endo's result using Khovanov-type homology or Gsvath-Jzabo-Rasmussen type techniques.

An approach to classical concordance in vts

Prop. If K is a slice link then

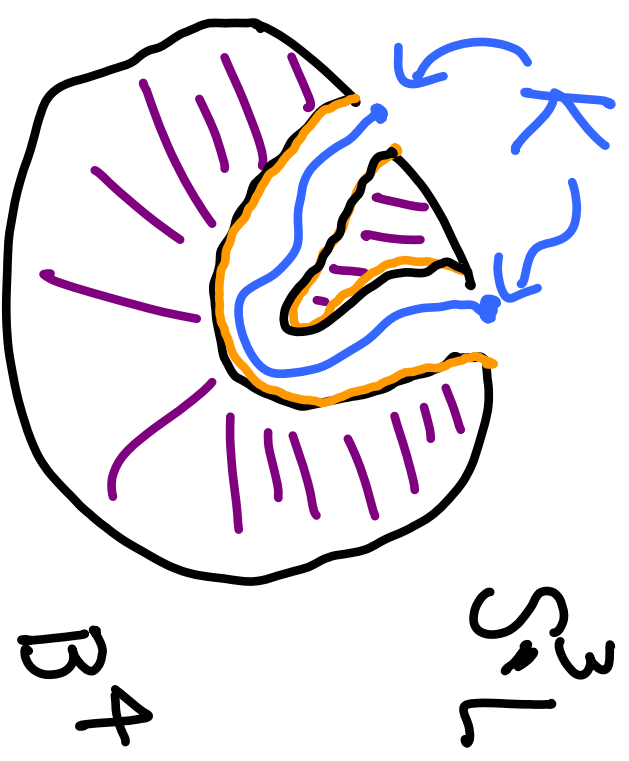
M_K , the 0-framed surgery on K , bounds a 4-manifold W such that

$$\cdot H_1(M_K) \xrightarrow{\cong} H_1(W) \cong \mathbb{Z}^m$$

$$\cdot H_2(W) = 0$$

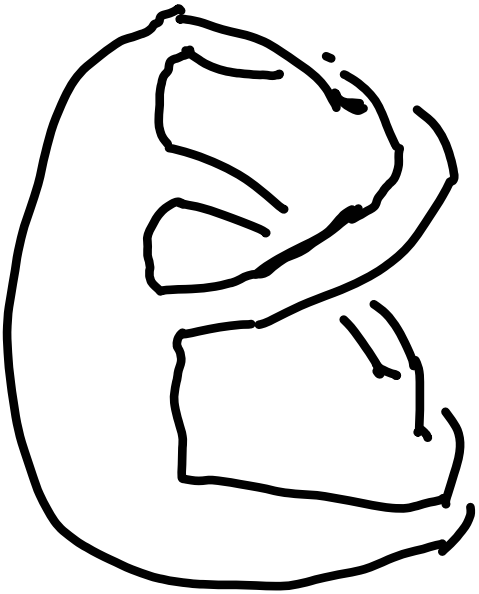
Proof: Let W be $B^4 - N(\text{slice disks})$

Exercise: almost if and only if



This observation leads to all classical invariants:

1. $M_k = 0$ in $\mathcal{J}_3^{\text{Spin}}(K(\mathbb{Z}^m, 1))$
 - Arf invariants
 - some Milnor invariants



$m=1$

FIGURE 8 KNOT

Arf Invariant $\neq 0$



$m=3$

Borromean Rings

does not bound

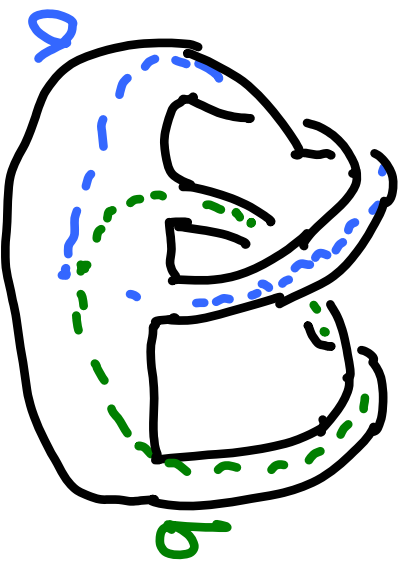
"over $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ "

2. The ∞ -cyclic cover of M_k is the boundary of the ∞ -cyclic cover of W^4 : \mathbb{Z}^m -cover for general m

• signature invariants associated to the equivariant intersection form on $H_2(W_\infty)$ must vanish since $H_2(W) = 0$.

(Lemma: $\Rightarrow H_2(W_\infty)$ is Torsion module)
 $m=1$

Levine - Tristram - Milnor signatures:



$$V = \text{Seifert matrix of } K \\ = (lk(a_i, b_j^+))$$

w complex number of norm 1

w -signature of $K = \text{signature}(wV + \bar{w}V^T)$

3. Equivariant Poincaré Duality for (W_∞, M_∞) implies “ $\frac{1}{2}$ -lives and $\frac{1}{2}$ dies” in

$$H_1(M_\infty; \mathbb{Q}) \rightarrow H_1(W_\infty; \mathbb{Q})$$

implying Alexander polynomial of a slice knot factors

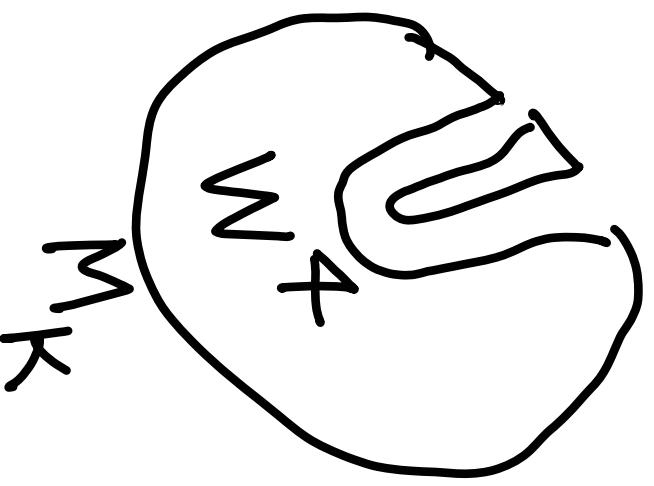
$$\Delta(t) = f(t)f(t^{-1})$$

So: $\Gamma_{\text{TOP}} \xrightarrow{\cong} \Sigma_{\infty} \times \Sigma_{\infty} \times \Sigma_{\infty} \times \Sigma_{\infty}$
 (Levine)

Kernel is called "algebraically slice knots"
 These are all the classical knot concordance invariants.

How to get "higher-order" obstructions?

Find other groups Γ that can complete diagram:



Then we can get higher-order concordance invariants associated to Γ -covering space

- higher-order Arf invariants (bordism)
- " " signatures (L^2 -signatures)
- " " Alexander Polynomials
- " " Alex. Modules ($H_1(\Gamma\text{-cover})$)

1973 Γ = metabelian Casson-Gordon

defined signatures associated to Γ

\Rightarrow {Algebraically Slice} $\cong \mathbb{Z}_1 \times \mathbb{Z}_2^\infty$

(Casson-Gordon, Jiang, Livingston)

'99 program works for $\Gamma = \text{torsion-free-solvable}$
group (C-Orr-Teichner)

~2002 $\Rightarrow \Sigma^\infty \subseteq \{ \text{KNOTS with vanishing?} \}$
CASSON - Gordon invariants
 $\Gamma = \pi_1(W) / \pi_1(W)^{(3)}$

Aside: Higher-order Alexander
modules used by Harvey, Cochran
are $\pi_{(n)}$ associated to iterated
 $\overline{\pi_{(n+i)}}$ universal abelian covers of M_K

These led to new invariants δ_n that are degrees of higher-order Alexander polynomials and give sharper estimates than usual Alexander polynomial for genus (K) or Thurston norm.

$$\delta_0 \leq \delta_1 \leq \delta_2 \leq \dots \leq 2 \text{genus}(K)$$

There should be a refinement of

Link FLOER Homology where these δ_n can be "read off", but

none of these degrees of Alex.
polynomials is a concordance invt.
So they will not appear today.

ANOTHER CLASSICAL SET OF CONCORDANCE INVTS:

A different crucial advance was taken in 1963 by John Stallings but applies only to links $m \geq 1$.

RECALL: $S^3, K \rightarrow W = B^4 - \Delta$ is K slice $\Rightarrow \exists W$ such that

a homology isomorphism.

Thm (Stallings) If $f: M \rightarrow W$ is

- isomorphism on $H_1(-; \mathbb{Z})$
- epimorphism on $H_2(-; \mathbb{Z})$

then A_n

$$\frac{\pi_1(M)_n}{\pi_1(M)_n} \cong \frac{\pi_1(W)}{\pi_1(W)_n}$$

$A_n = n^{\text{th}}$ term of lower central series
of group A

$$A_1 = A, \quad A_{n+1} \cong [A, A_n].$$

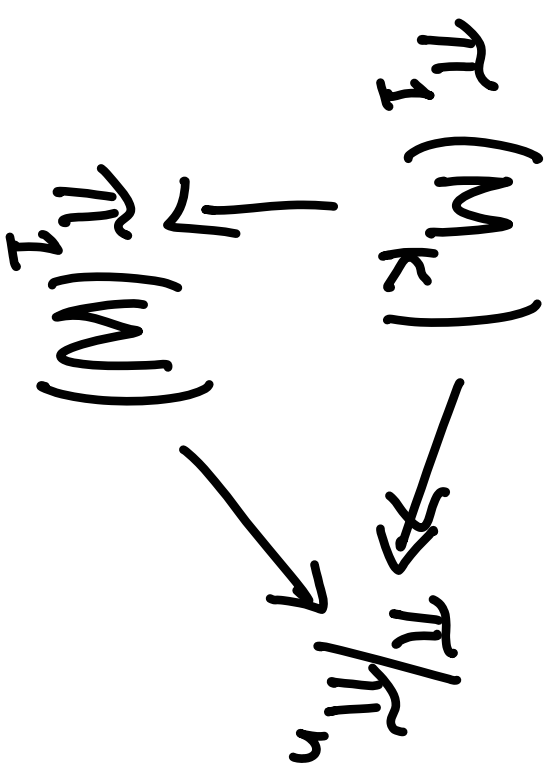
Consequently: If K is a slice link

$$\text{Then } \frac{\pi_1(S^3, K)}{\pi_1(S^3, K)_n} \cong \frac{F\langle X_1, \dots, X_n \rangle}{F_n} A_n$$

and longitudes $\in \pi_1(S^3, K)_n$ A_n ,

so Milnor's $\bar{\mu}$ -invariants are
concordance invariants.

Also Can now use $\Gamma = \pi/\pi_n$
 to get signature
 obstructions and
 ranks of Alexander
 modules associated
 to this Γ



(Smolinsky)

THIS CONCLUDES MY INTRO TO LINK
 CONCORDANCE INVARIANTS

Other Equivalence Relations

- give more information about structure in $\mathcal{C}_0^{\text{TOP}}$ and $\mathcal{C}_0^{\text{smooth}}$

Idea: commutator series **filter** a group G

$$\dots \subseteq G_3 \subseteq G_2 \subseteq G$$

and to each commutator series there is a **family of equivalence relations** **filtering** the link concordance group.

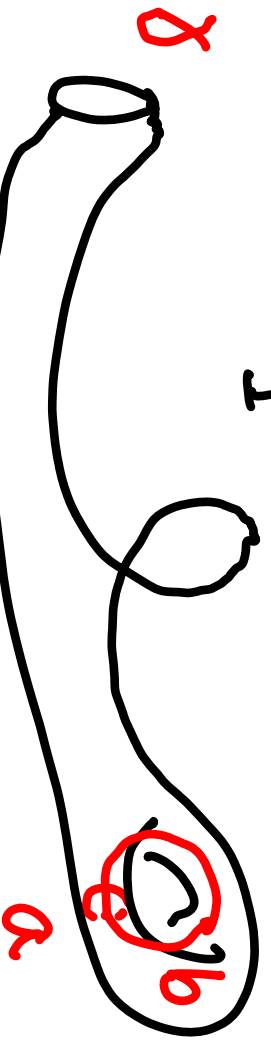
Derived Series of G : $G^{(0)} \equiv G$ $G^{(1)} = G' = [G, G]$

$$G^{(n+1)} \equiv [G^{(n)}, G^{(n)}]$$

there are other commutator series.

Basic Idea: a circle γ is a commutator

$$\gamma = [a, b] \text{ in } \pi_1(X) \iff \gamma \text{ bounds a}$$



(NOT embedded)

Similarly $\gamma = [[c, d], [e, f]]$ in $\pi_1(X)$

$\iff \gamma$ bounds a surface



$$\gamma = [[c, d], [e, f]]$$

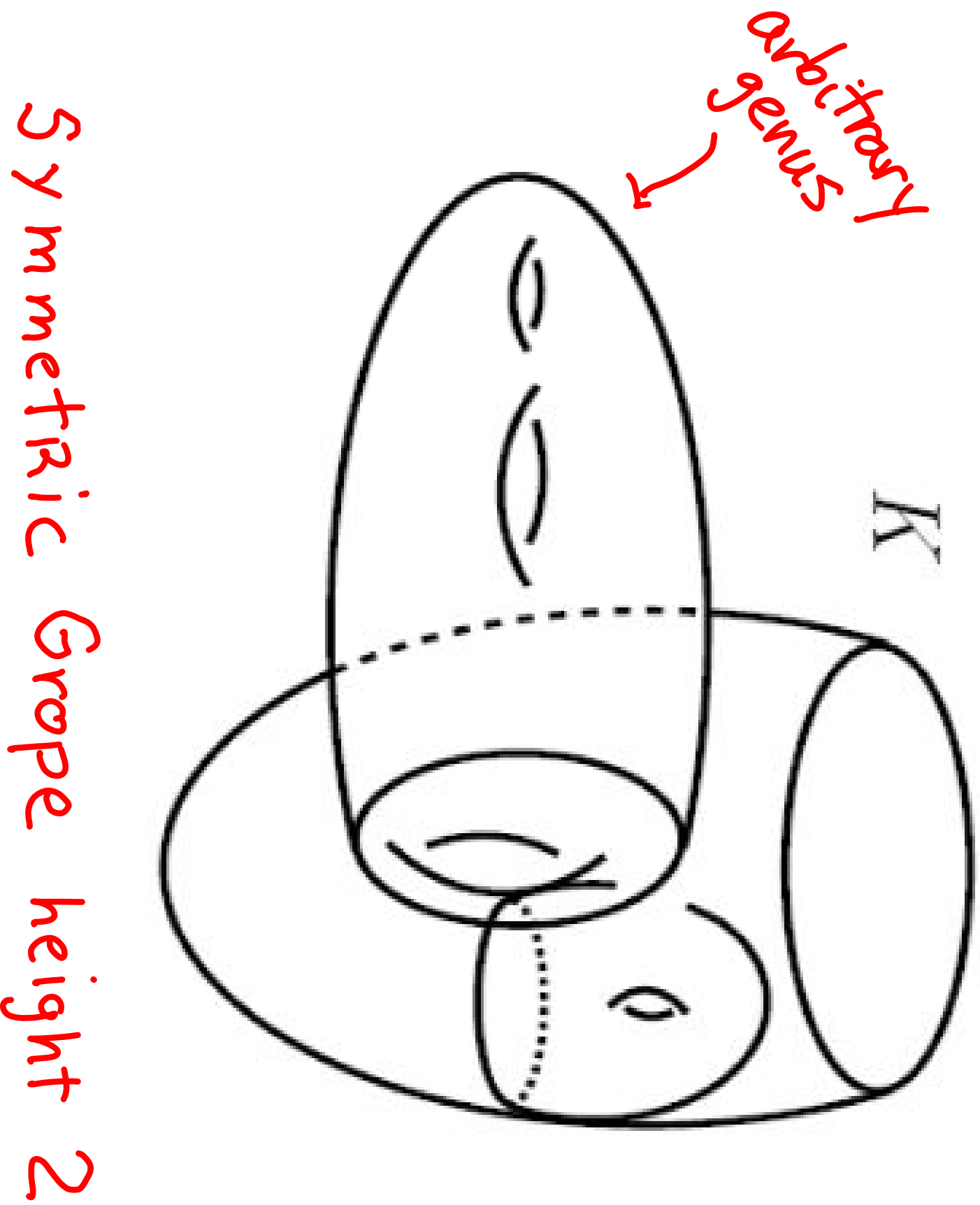
We call such 2-complexes consisting of layers of surfaces glued to elements of a symplectic basis of a previous surfaces:

Gropes (different type for each series)

Symmetric or full-grope corresponds to derived series : attach surface to all elements of symplectic basis of previous surfaces.

HALF - Grope : corresponds to lower central series : attach surfaces to $1/2$ of symplectic basis.

Which links bound embedded gropes in S^3 or B^4 ?



Symmetric

- Knots that cobound, with the unknot, **half-gropes** in S^3 with n -stages are those knots with vanishing Vassiliev invariants of order $\leq n$ ($\neq 1$) (Conant-Teichner, Habiro)
 - links that bound **$\frac{1}{2}$ -gropes** in B^4 are intimately related to Milnor's $\bar{\mu}$ -invariants (Cochran, Krushkal-Teichner)
- We focus on symmetric gropes related to derived series

Let $\mathcal{G}_n^m =$ concordance classes of m -comp. links that bound embedded height n symmetric gropes in B^4

$$\{e\} \subseteq \dots \subseteq \mathcal{G}_2^m \subseteq \mathcal{G}_1^m \subseteq \mathcal{C}_m^m$$

SLICE LINKS

Previous RESULTS on KNOTS $m=1$ (C-Dr-Teichner)

$K \in \mathcal{G}_3 \Rightarrow K$ is algebraically slice

$K \in \mathcal{G}_4 \Rightarrow K$ has vanishing Casson-Gordon

$$\Rightarrow \mathbb{Z}_1^\infty \subseteq \mathcal{G}_4 / \mathcal{G}_5$$

• $\mathcal{G}_n / \mathcal{G}_{n+1}$ is infinite $\forall n \geq 2$ (C-Teichner)

• For any knot K with degree $\Delta_K(t) \geq 2$, and any $n \geq 2$ there is an infinite set of knots each \mathcal{N}_n -smoothly n -gropo cobordant to K but all distinct up to $(n+1)$ -gropo cobordism and all with the same S and τ invariants (O-S (Rasmussen)).

(C-T. Kim)

(examples have same genus so all $S, \tau \in \{-2g, \dots, 0, \dots, 2g\}$)

RESULTS FOR LINKS: $m \geq 2, n \geq 1$

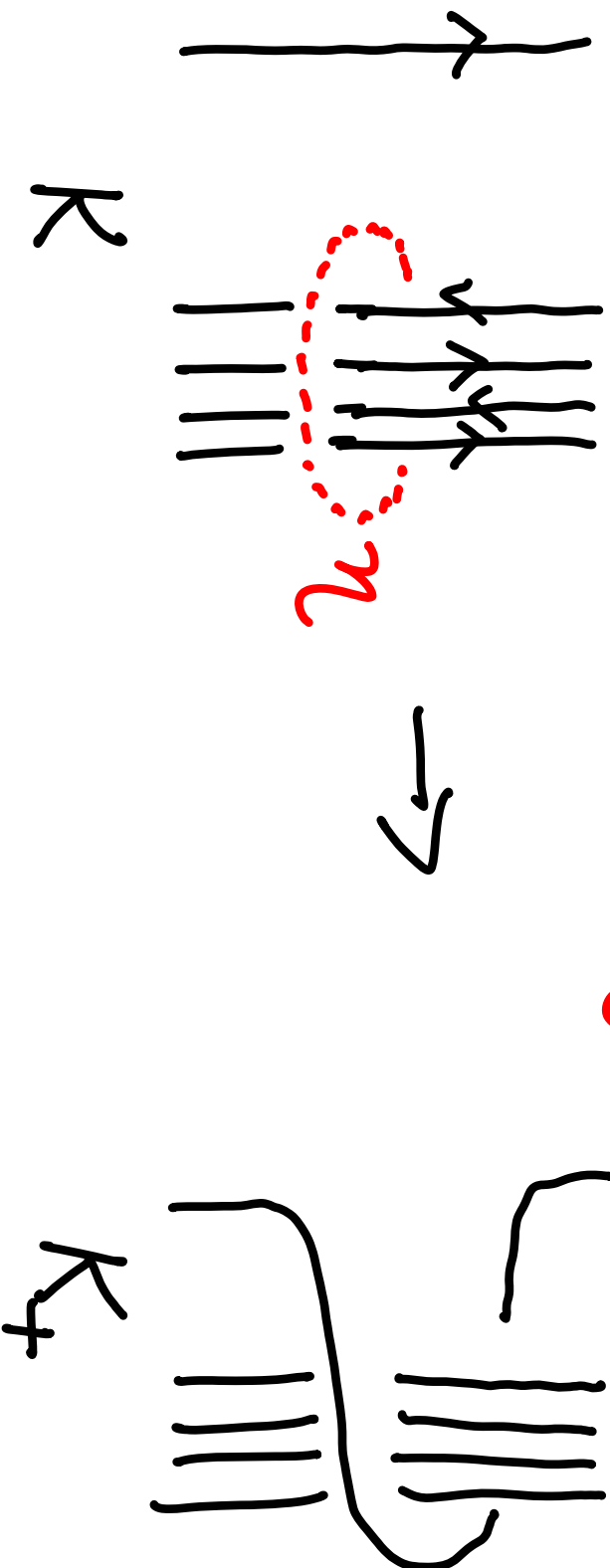
- $\mathcal{L}_n / \mathcal{L}_{n+2}$ contains a subgroup consisting entirely of boundary links that maps onto \mathbb{Z}_∞ . (Harvey)

- Milnor's invariants give maps

$$\mathcal{L}_n / \mathcal{L}_{n+1} \longrightarrow \underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_{\text{some finite } \#}$$

How to create knots and links that bound height n symmetric gropes in B^4 (C-Teichner)

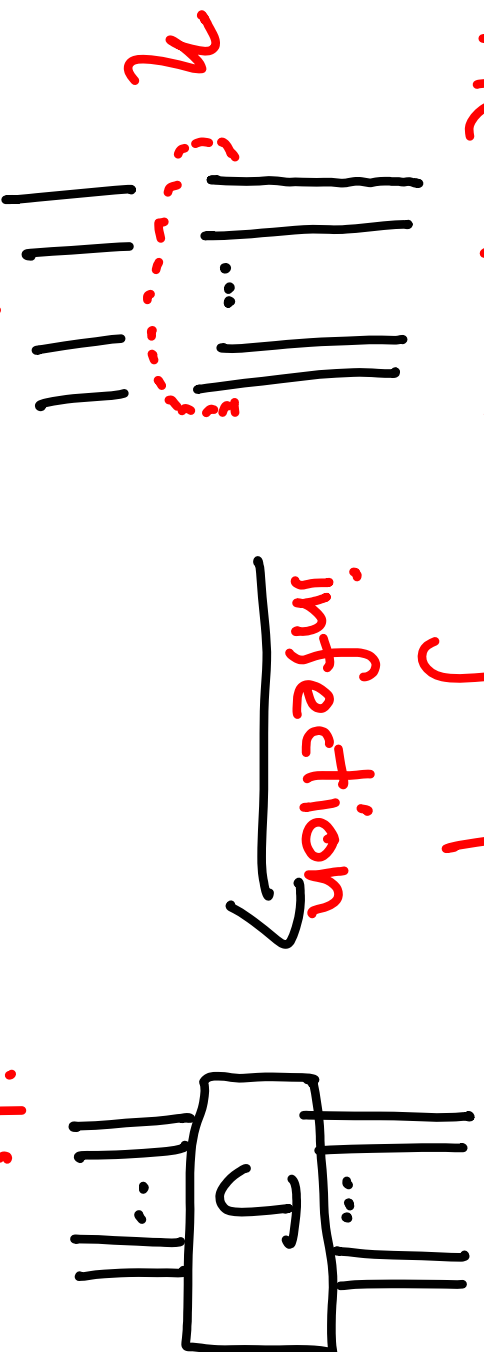
Generalized Crossing Change



* Suppose η bounds embedded n -grope in S^3, K . (in particular $\eta \in \pi_1^{(n)} \mathbb{Z}^{n-1}$, clasper

Prop. 5.1 K and K' are n -gropes cobordant
 so if K was trivial or a slice link then
 K' bounds embedded n -gropes in B^4 .

Corollary: The same is true for
 the following operation



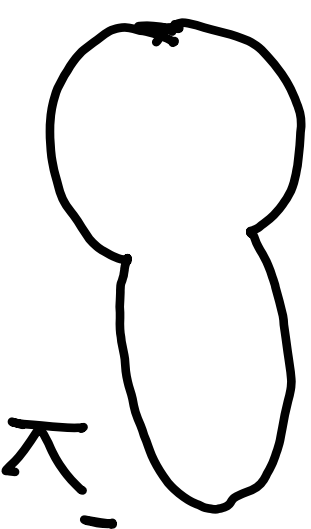
since it is a composition of
 generalized crossing changes!

Proof that K and K' are n -gropes cob.

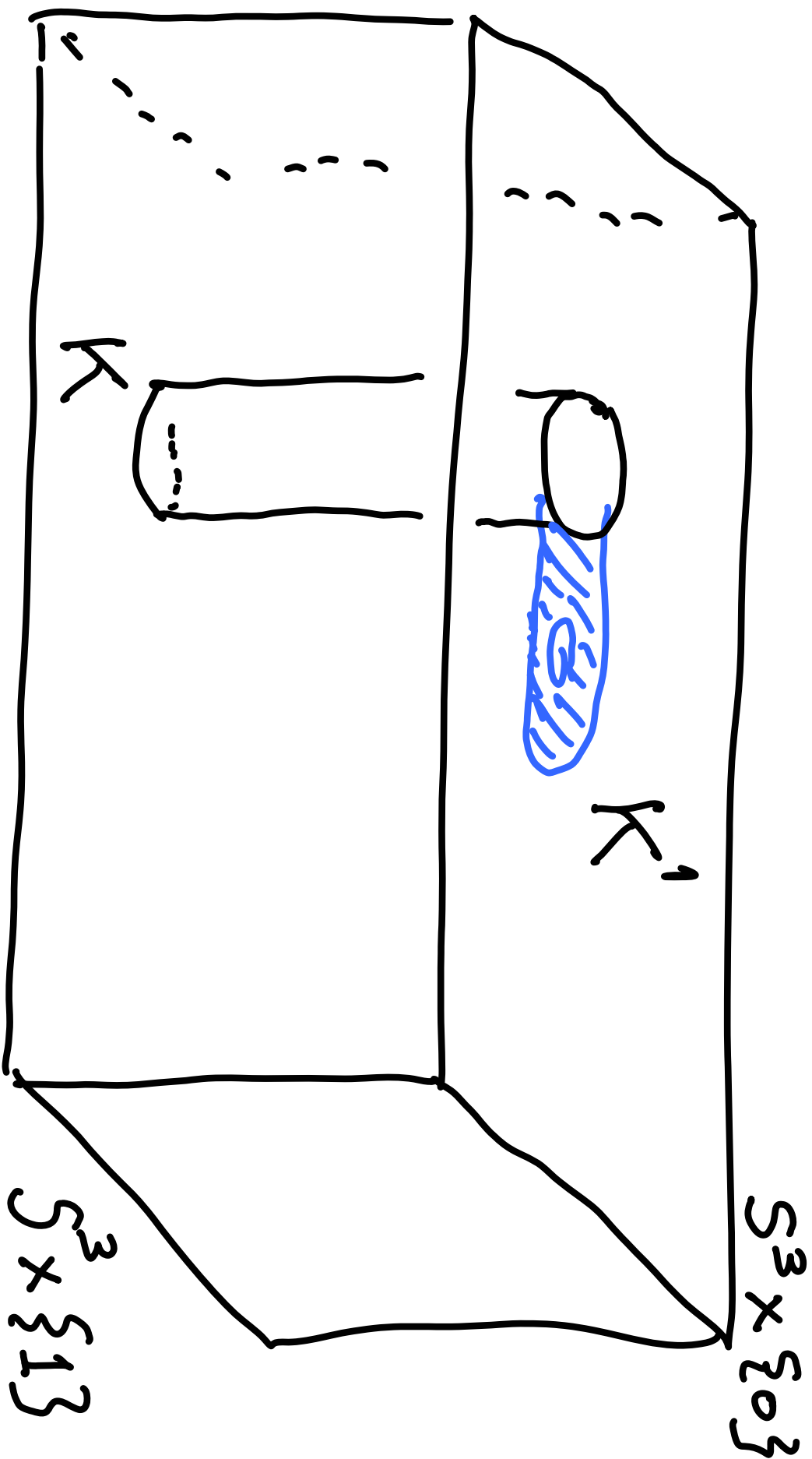
1. Start with product concordance

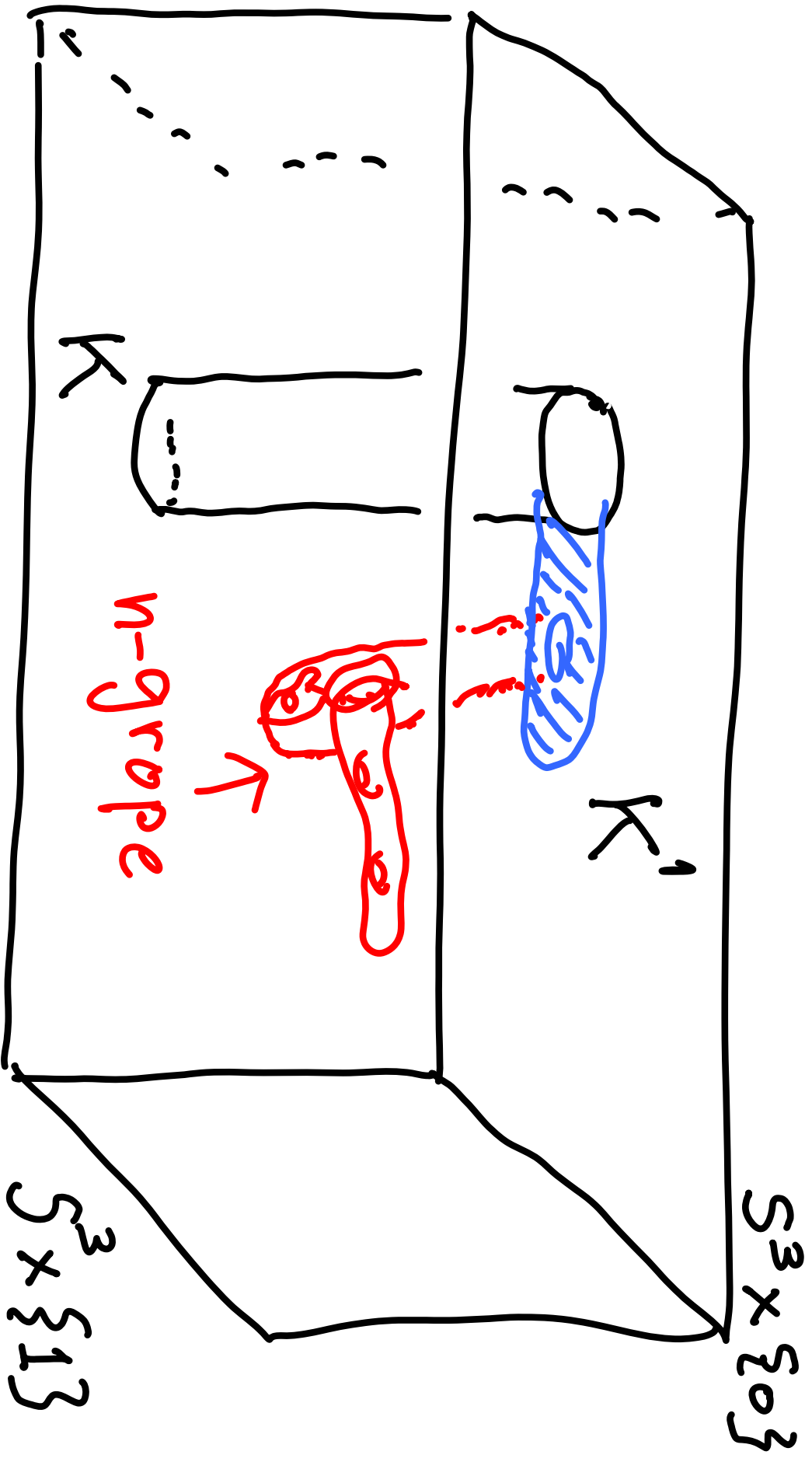
$$K \times [0, 1] \subseteq S^3 \times [0, 1]$$

2. Add Blue punctured disk to $K \times \{0\}$

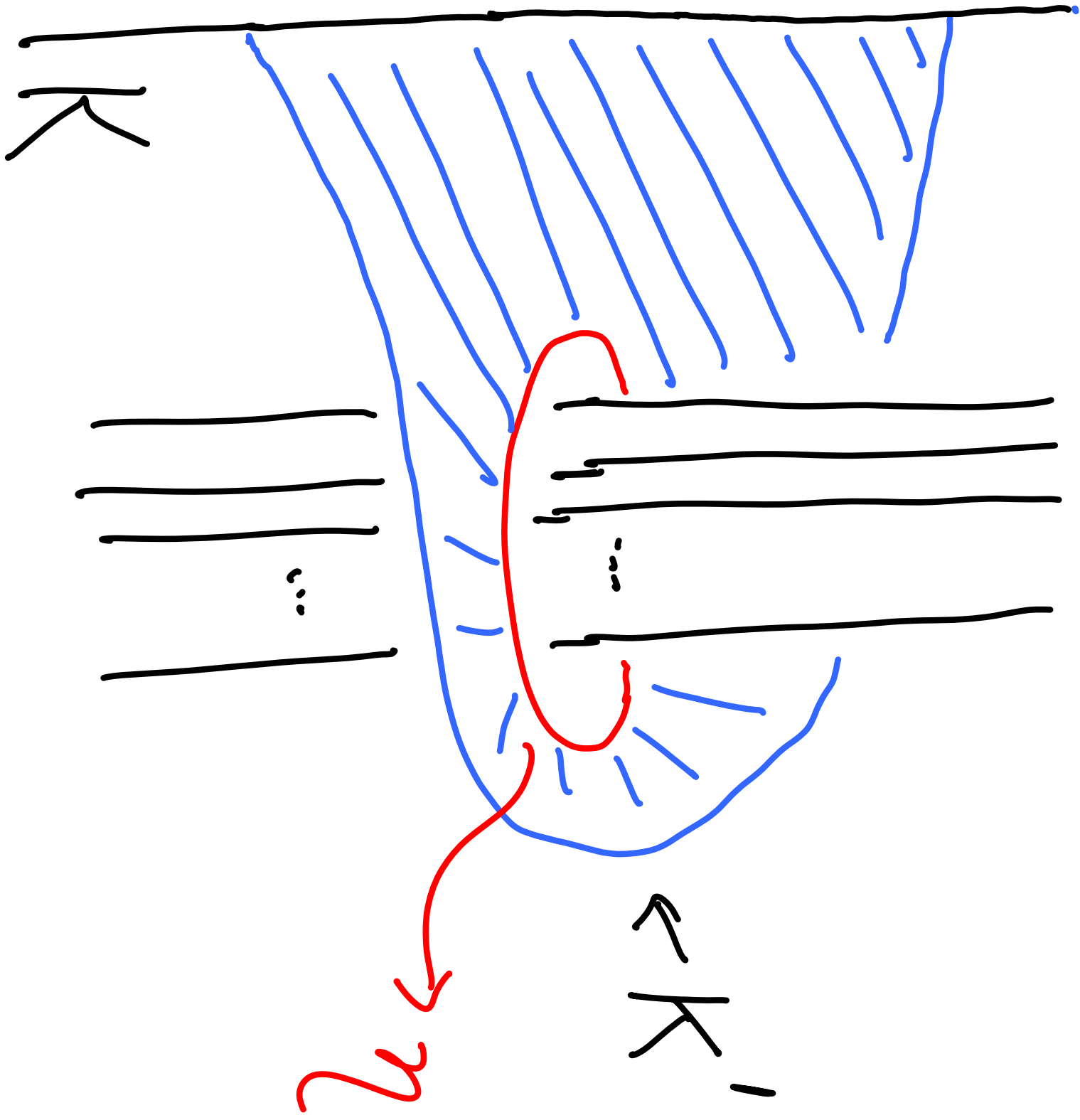


such that puncture is η
and "new boundary" is K'

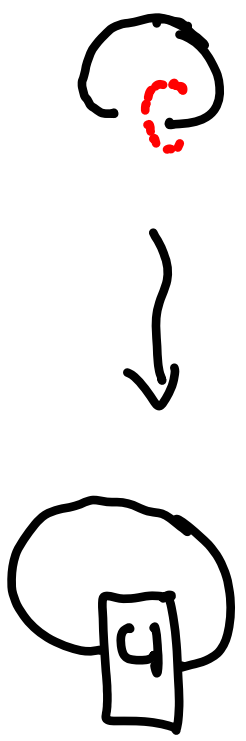




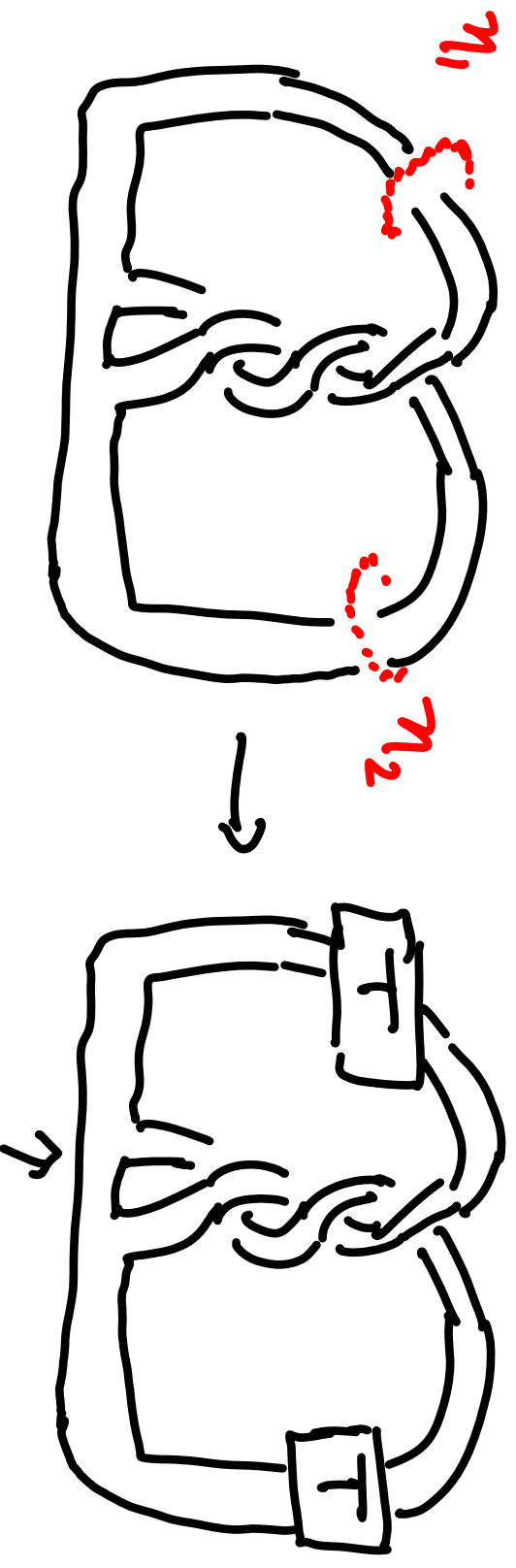
Result is n -grope cobordism
 from K' to K



Examples: $n=0$

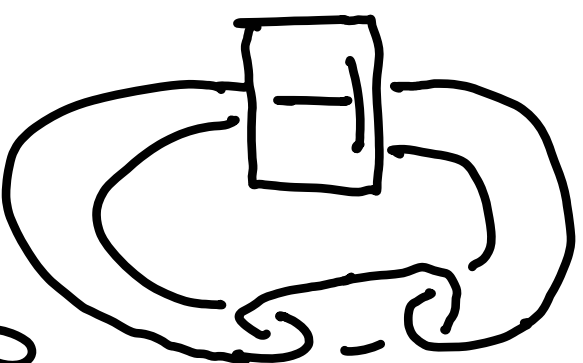
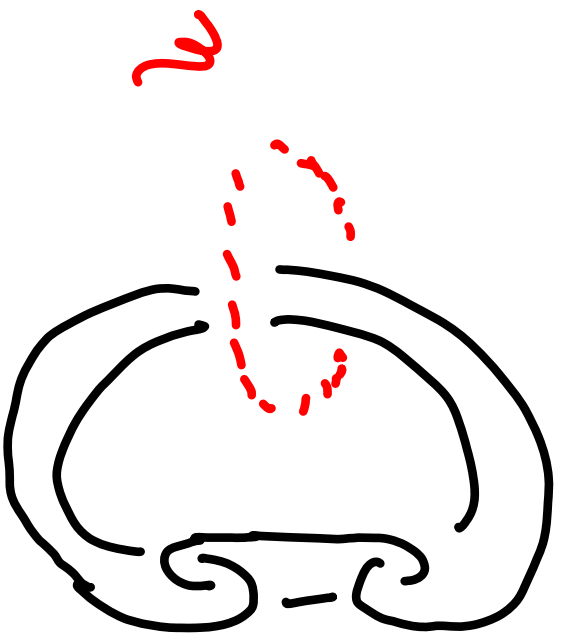


$n=1$



doesn't bound
a 4-grope by higher-order signatures

$n=1$



Bing Double
of Trefoil

not in \mathcal{Z}_4 by
higher-order signature invariant
(Harvey)

For arbitrary $n \geq 0$ it is always possible to find such η for trivial link $(n > 1)$



since $\Pi_1 = \text{Free group}$ and $\bigcap_{n=0}^{\infty} F^{(n)} = \{e\}$

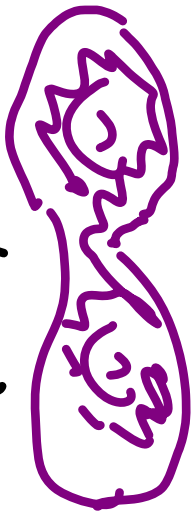
Infecting by trefoil produces link that bounds an n -groppe by Proposition but doesn't bound $(n+3)$ -groppe (Harvey) in particular is not slice.

For any slice knot K , $\Delta_K(t) \neq 1$,

the same can be done

Choose X, Y independent in Alex.

module of K :



choose

$\eta \in F^{(n)} - F^{(n+1)}$

Prop. (C-): $n \in G^{(n+1)} - G^{(n+2)}$ $G = \Pi_1(S^3 - K)$

This is the way one produces links that are non-trivial at each stage of the Grope filtration ($m=1$ C-Teichner, $m>1$ Harvey)

But why are there invariants that obstruct bounding long gropes?

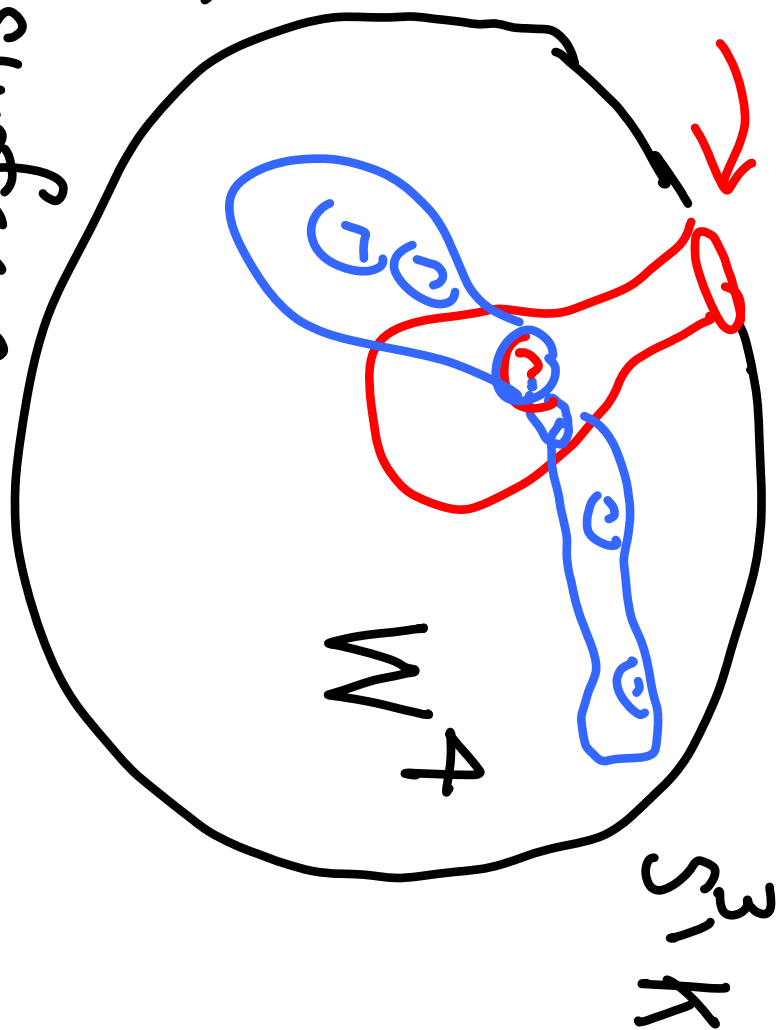
Basic Idea:

K

Suppose K bounds

an $(n+2)$ -groppe

in B^4 . Consider

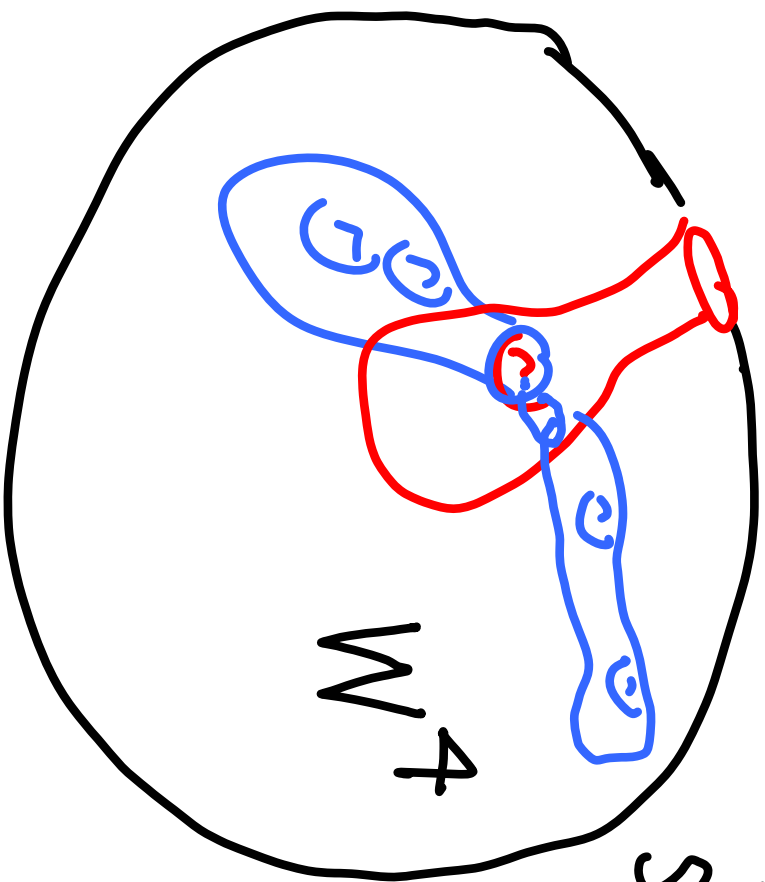


$W^4 = B^4$ - first surfaces

then $H_1(S^3, K) \cong H_1(W) \cong \mathbb{Z}^n$

as before but now $H_2(W) \neq 0$,

so signatures may be $\neq 0$, No Stallings
Thm.



Key: $H_2(W^4)$
 S^3 is generated
 by n -surfaces

$$\pi_1(\Sigma) \subseteq \pi_1(W)^{(n)} = \Sigma$$

Is there an analog of Stallings' theorem for derived series and with H_2 hypothesis weakened?

with more work one can show if

K bounds $(n+2)$ -gropes then

$M_K = \partial W^4$ such that

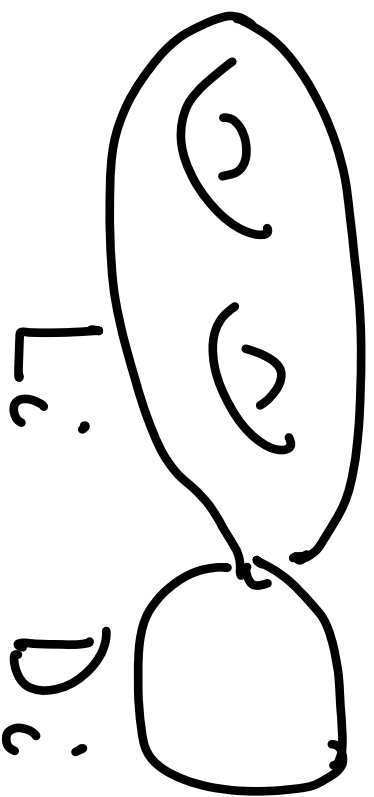
$$\bullet H_1(M) \cong H_1(W)$$

$\bullet H_2(W)$ is generated by

embedded n -surfaces L_i, D_i

that intersect

$$\text{so } \bigoplus (\circlearrowleft \circlearrowleft)$$



Thm (Stallings) If $f: M \rightarrow W$ is

- isomorphism on $H_1(-; \mathbb{Z})$
- epimorphism on $H_2(-; \mathbb{Z})$

then A_n

$$\frac{\pi_1(M)_n}{\pi_1(M)_n} \cong \frac{\pi_1(W)}{\pi_1(W)_n}$$

There is an analogue of Stallings' theorem for derived series

and the hypothesis on H_2 can be weakened to what we need!

$G_H^{(n)} = \text{torsion-free derived series}$
(Harvey)

Theorem (C-Harvey) $f: M \rightarrow N$

- monomorphism $H_1(M; \mathbb{Q}) \rightarrow H_1(N; \mathbb{Q})$
- epimorphism $H_2(M; \mathbb{Q}) \twoheadrightarrow H_2(N; \mathbb{Q})$
<n-surfaces>

then $\pi_1(M) \rightarrow \pi_1(N)$

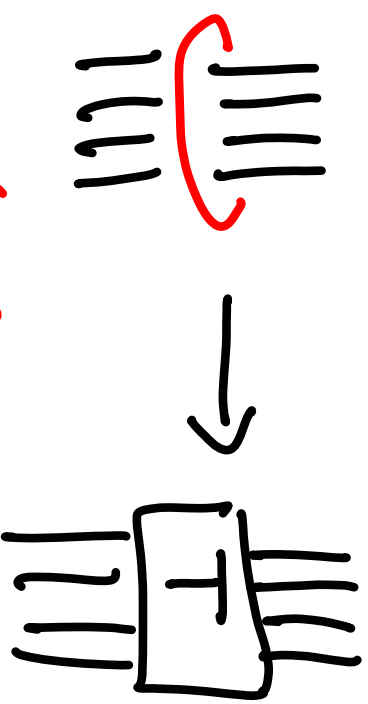
$$\frac{\pi_1(M)^{(n+1)}_H}{\pi_1(N)^{(n+1)}_H}$$

(Dwyer)

Why are such injectivity results crucial? Recall we chose

$$\eta \in \frac{\pi(S^3 \setminus K)^{(n)}}{\pi(S^3 \setminus K)^{(n+1)}} \quad \text{non-zero and}$$

did "infection"



but REALLY NEED

$$i_*(\eta) \neq 0$$

$$\frac{\pi_1(S^3 \setminus K)^{(n)}}{\pi_1(S^3 \setminus K)^{(n+1)}} \xrightarrow{i_*} \frac{\pi_1(W)^{(n)}}{\pi_1(W)^{(n+1)}}$$

Applications:

- show ranks of certain higher-order Alexander modules are invariants of n -gropes cobordism

- improve on Harvey's results on grope filtration of link concordance
- show some results of

COT work for non-manifolds