Knot Concordance and Homology Cobordism

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$$\frac{\text{KNOTS}^{\ \text{in } S^3}}{\text{ISOTOPY}} \xrightarrow{\text{“surgery”}} \frac{\text{CLOSED 3-MFD}}{\text{HOMEO}}$$

$S(0,K) =$ zero surgery $=$ Green curve is attached to the longitude

Are these functions injective?

1. PROPERTY R: If $S(0,K) = S(0, \text{unknot } U)$ does $K = U$?

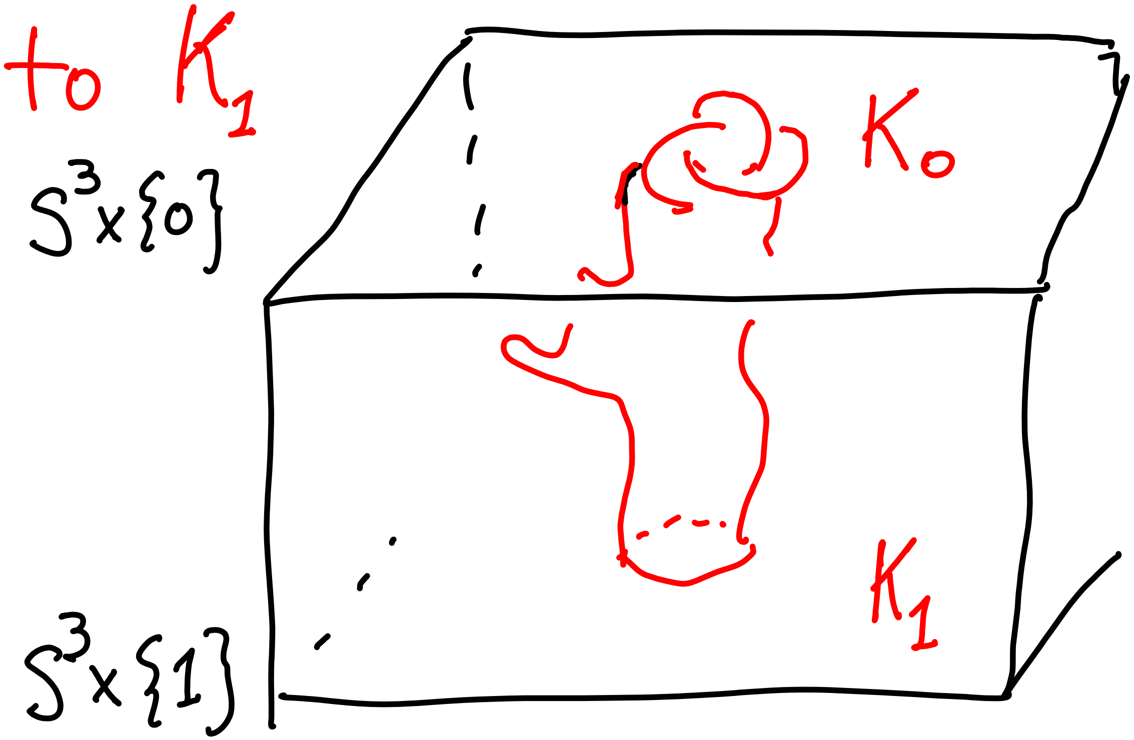
2. Property P: If $S(\pm 1, K) = S(\pm 1, U) = S^3$ does $K = U$?

3. When does surgery on K yield the same as surgery on U = LENS SPACE?

Natural 4-dimensional version of this

$$\frac{\text{KNOTS}}{\text{concordance}} \xmapsto{\text{surgery}} \frac{\text{closed 3-mfds}}{\text{homology cobordism}}$$

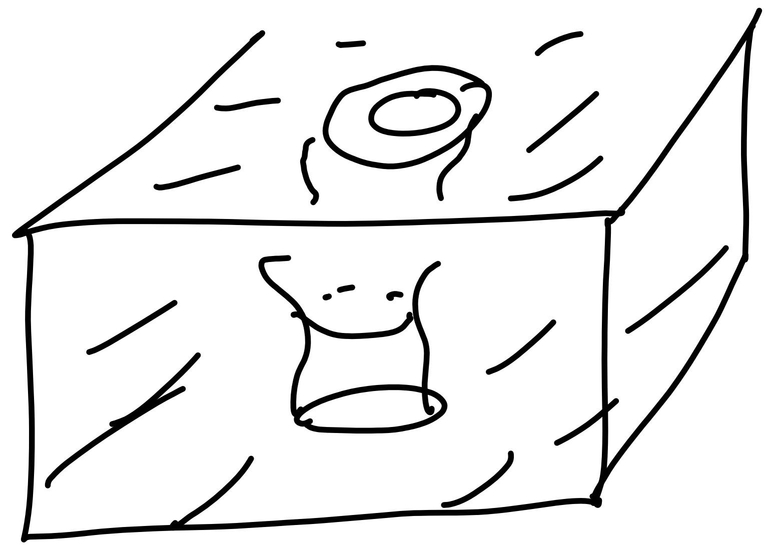
K_0 is concordant to K_1 if they cobound a flat annulus in $S^3 \times [0,1]$



Two 3-mfds M_0, M_1 are homology cobordant if $\exists$ 4-mfd W^4 $\partial W^4 = M_0 \amalg -M_1$ and $H_*(M_i) \to H_*(W)$ are isomorphisms.

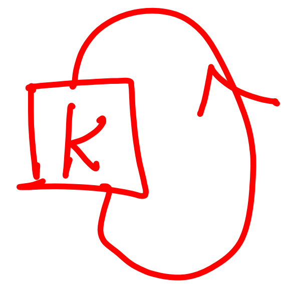
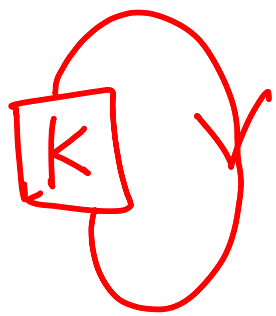
Why is the surgery map well-defined?

Alexander DUALITY $\Rightarrow$ exterior of annulus is a homology product $\Rightarrow$ same for any Dehn surgery



4-DIM PROPERTY R

<u>Question</u>: If $S(0,K)$ is smoothly homology cobordant to $S(0,K')$ then is K smoothly concordant to K' as unoriented knots.

<u>Why unoriented?</u> A knot  and its REVERSE  have same exterior and same 0-surgery but are not generally isotopic nor concordant.

<u>Def:</u> A homology cobordism W between $S(0,K)$ and $S(0,K')$ is rel meridians if the inclusion induced

$$
\begin{array}{ccccc}
H_1(S(0,K)) & \longrightarrow & H_1(W) & \longleftarrow & H_1(S(0,K')) \\
\text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\
\mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \xleftarrow{\cong} & \mathbb{Z} \\
 & & & & \\
\mu & & \longleftrightarrow & & \mu'
\end{array}
$$

sends positive meridian of K to positive meridian of K'.

Also can ask that $\pi_1(W)$ be normally generated by μ

Evidence for a YES ANSWER:

1. YES if one of knots is unknot in TOP category, or smooth if 4-Dim Poincaré Conjectur

2. The homology cobordism type of knot exteriors does determine smooth concordance type if 4-D PC True.

Our Theorem: No! There exist Knots whose zero Surgeries a smoothly homology cobordant but which are not smoothly concordant. There are examples that are topologically concordant to trivial Knot.

We also show that a natural $\mathbb{Q}$-version

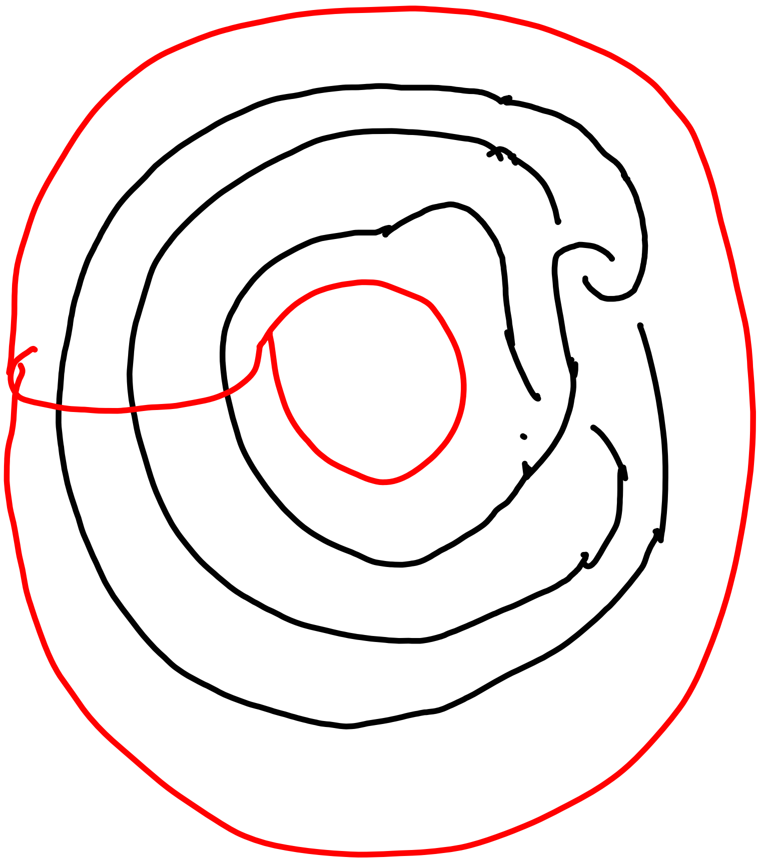
$$\frac{\text{Knots}}{\text{rational concordance}} \xrightarrow{\text{0-surgery}} \frac{\text{3-mfds}}{\mathbb{Q}\text{-hom. cobordism}}$$

is not injective even in TOP category

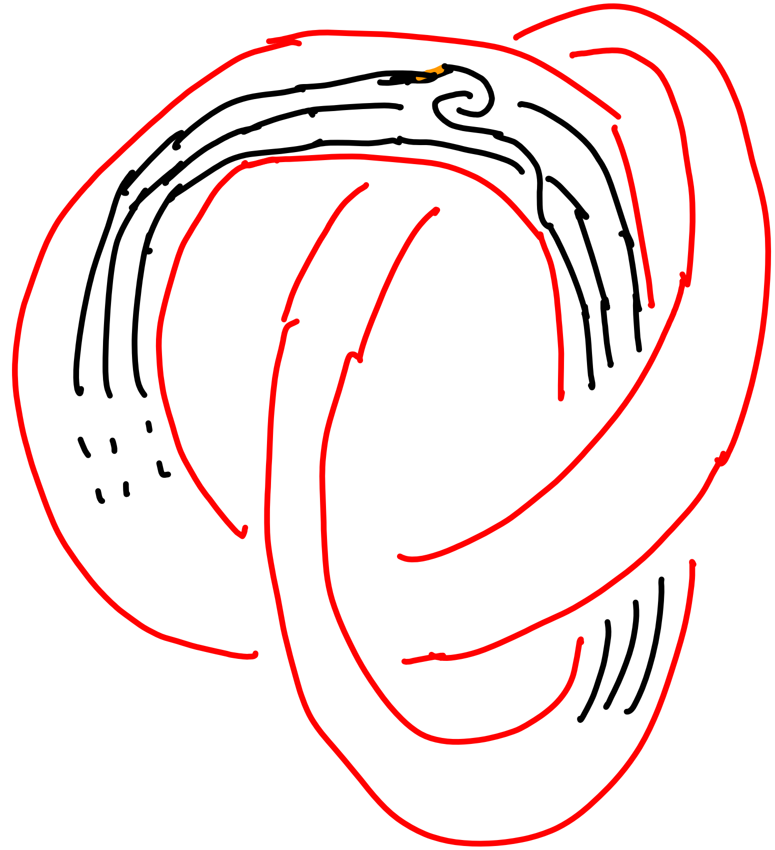
The examples :

Consider K and a satellite knot $P(K)$ of winding number 1. We show the 0-surgeries are homology cobordant but usually K is not concordant to $P(K)$ (uses τ, Heegard-Floer homology, contact topology).

Satellite Construction



tie in knot
K

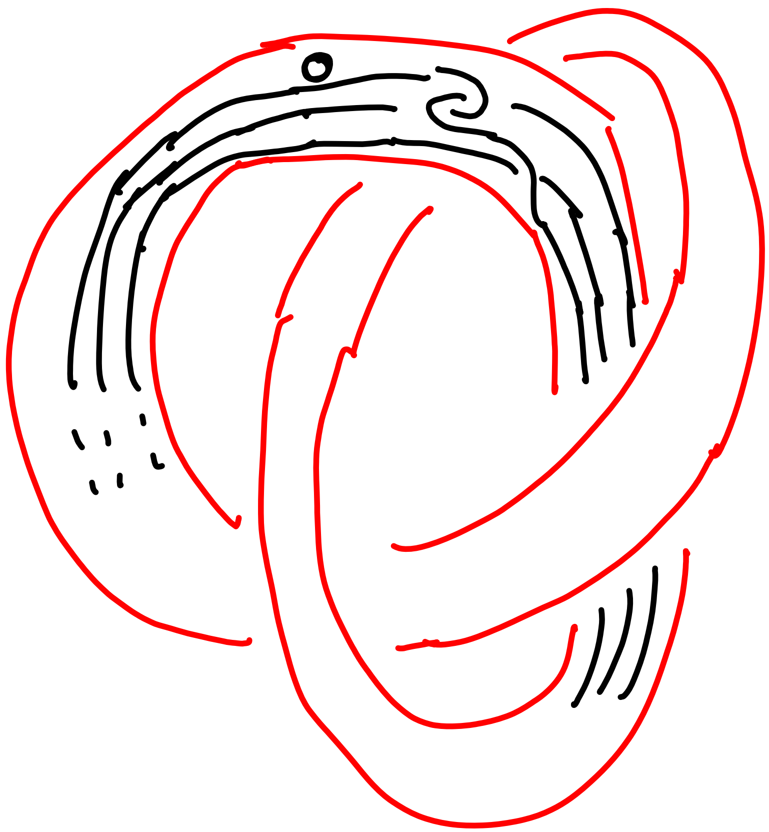


Pattern Knot
P in solid torus

P(K)

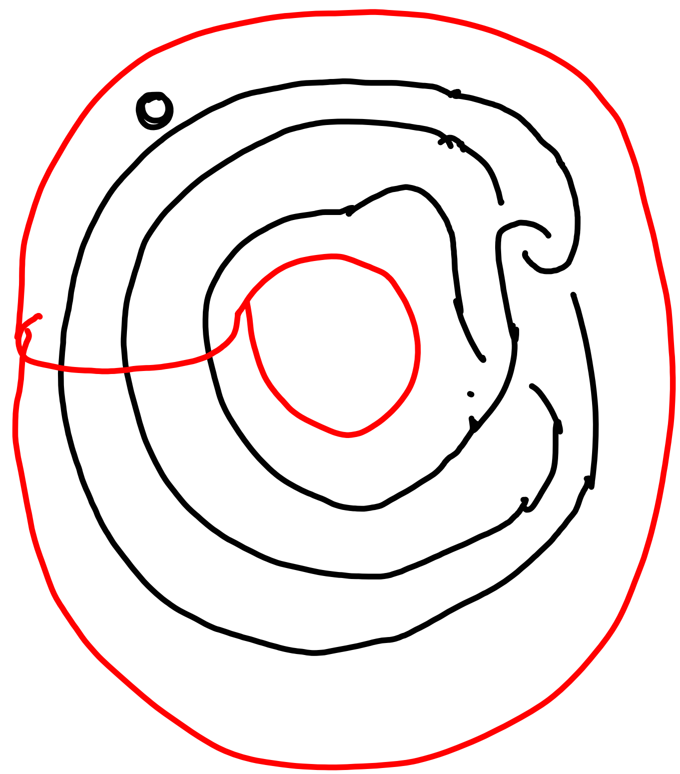
CLAIM: S(0, P(K)) homology cobordant to S(0, K)
if pattern is "unknotted in S^3".

Proof:

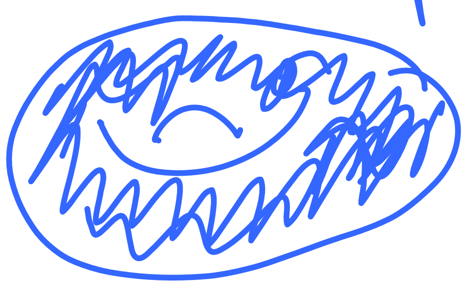
$S(0, P(K)) \cong$ 

DECOMPOSES INTO 2 PIECES:

A = OUTSIDE $= S^3 \setminus N(K)$

B = inside = 

Define a 3rd piece:

C = 

We have seen:

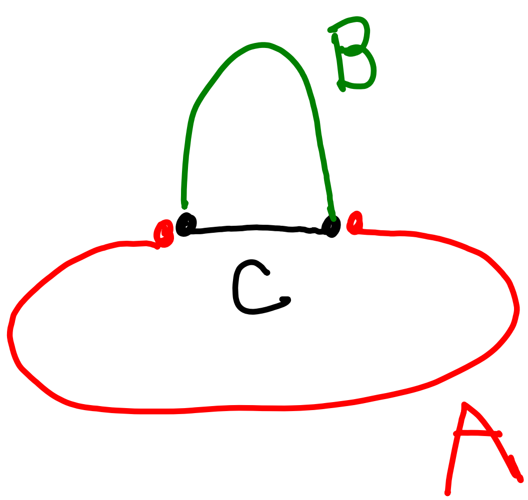
$$A \cup B = S(0, P(K))$$

$$B \cup C = S(0, P(U)) = S(0, U) \cong S^1 \times S^2$$

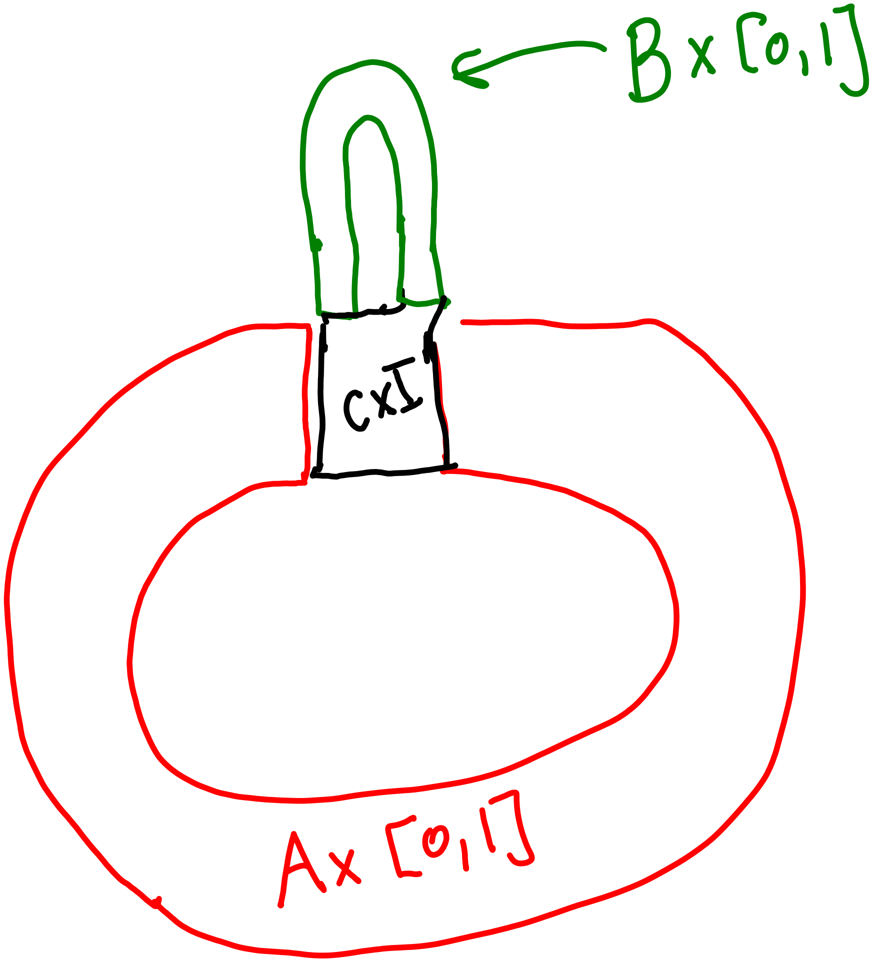
$$A \cup C = S^3 \setminus N(K) \cup \text{SOLID TORUS} \cong S(0, K).$$

Claim: $\exists$ cobordism between these 3:

Idea:



Thicken it up!



This is a 4-mfd with boundary

$$A \cup C \equiv S(0,K)$$

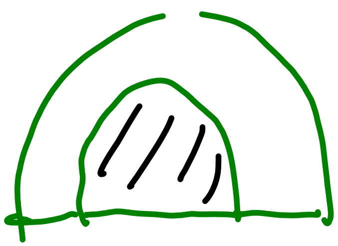
$$\|$$

$$-(A \cup B) \equiv -S(0,P(K))$$

$$\|$$

$$B \cup C \equiv S^1 \times S^2$$

Cap off $S^1 \times S^2$ with $S^1 \times B^3$ to get cobordism



between $S(0,K)$ $S(0,P(K))$.