The Seifert surface in Knot concordance

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Slice Knot K in S^3:

$$K = \partial\left(\Delta^2 \hookrightarrow B^4\right)$$

where Δ is smoothly embedded 2-disk.

Seifert surface for $K \hookrightarrow S^3$: orientable surface $F \hookrightarrow S^3$, $\partial F = K$.

- any knot has an infinite number of Seifert surfaces
- $\exists$ analogues of definitions for $S^{2n-1} \hookrightarrow S^{2n+1}$

Question: Can you use any Seifert surface to answer "Is K a slice knot?" Can you use circles on the Seifert surface to study this question?

Theorem (1969 J. Levine) for simple knot $S^{2n-1} \hookrightarrow S^{2n+1}$, $n>1$, linking numbers between cycles on "Seifert surface" determine if K is slice knot.

Suppose $K = \partial F \qquad F \hookrightarrow S^3$

Seifert Matrix of K: choose basis $a_1, \ldots, a_{2g}$ of $H_1(F)$, let $V_{ij} = lk(a_i, a_j^+)$

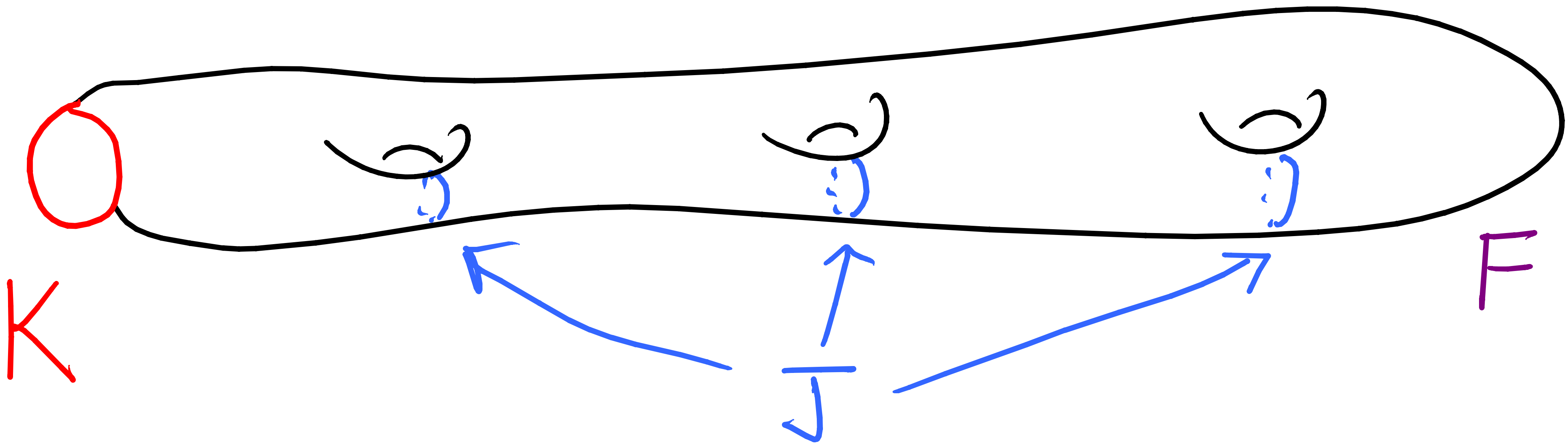
K is algebraically slice: there is some basis so that

$$V = \begin{array}{cc} \begin{array}{cc} g & g \end{array} & \\ \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} & \begin{array}{c} g \\ g \end{array} \end{array}$$

$lk(a_i, a_j) = 0 \qquad 1 \le i, j \le g$

These can be realized as embedded circles

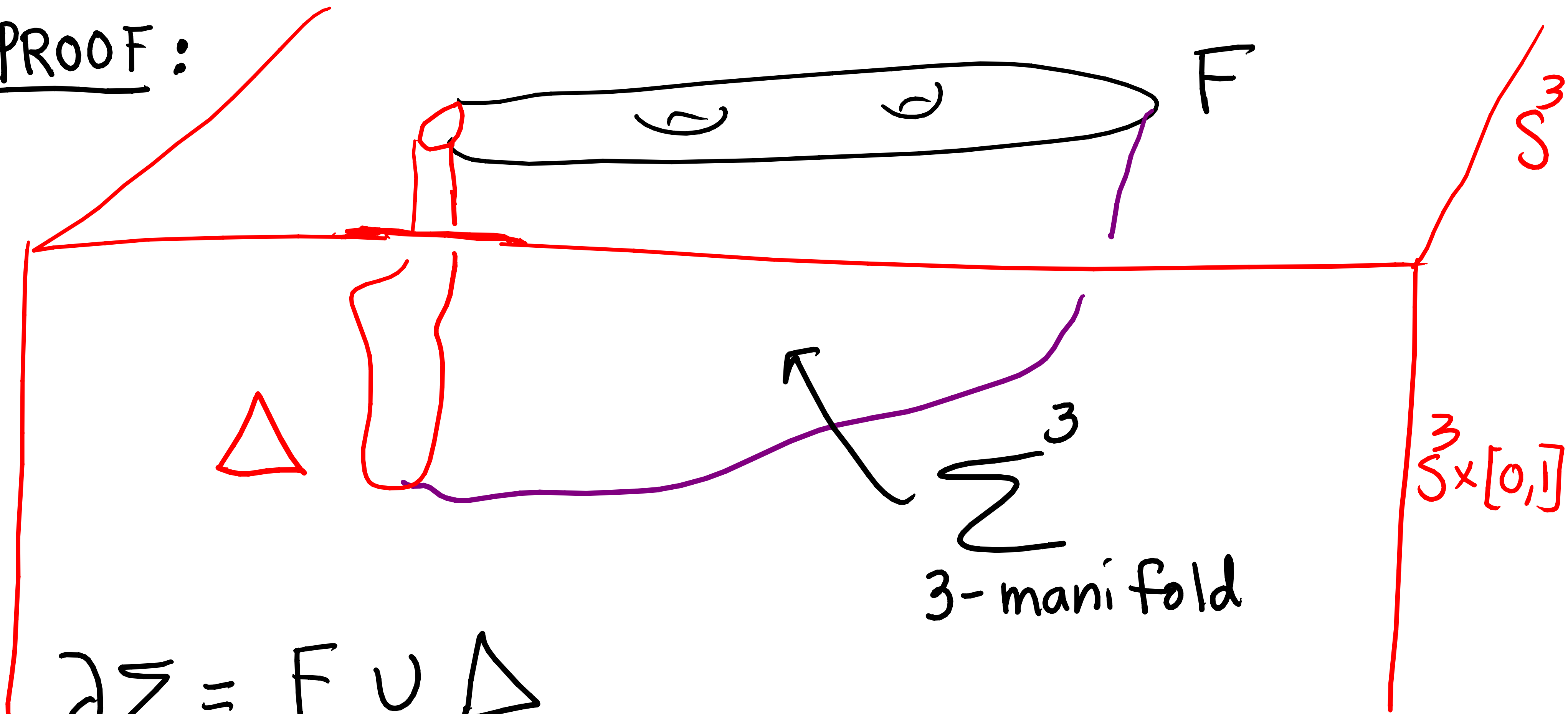
Derivative of K: a link $J = \{J_1, \dots J_g\}$ embedded on F, $lk(J_i, J_j) = 0$.



K algebraically slice $\Longleftrightarrow$ K has derivative J on any Seifert surface

Lemma (Levine) SLICE $\Longrightarrow$ ALGebraically slice, to any slice disk Δ and Seifert surface F there is an associated derivative link J.

PROOF:



$\partial \Sigma = F \cup \Delta$

FACT: Kernel $H_1(F;\mathbb{Q}) \longrightarrow H_1(\Sigma;\mathbb{Q})$ has rank g.

$\Rightarrow$ get $\{a_1, .., a_g\}$ s.t. $\mathrm{lk}(a_i, a_j^+) = 0$.

$\Rightarrow$ derivative link J associated to Δ.

Why are derivative links important?

1. Invariants of K expressed as SIMPLER invariants of J.

Example: Levine-Tristram signatures of K:

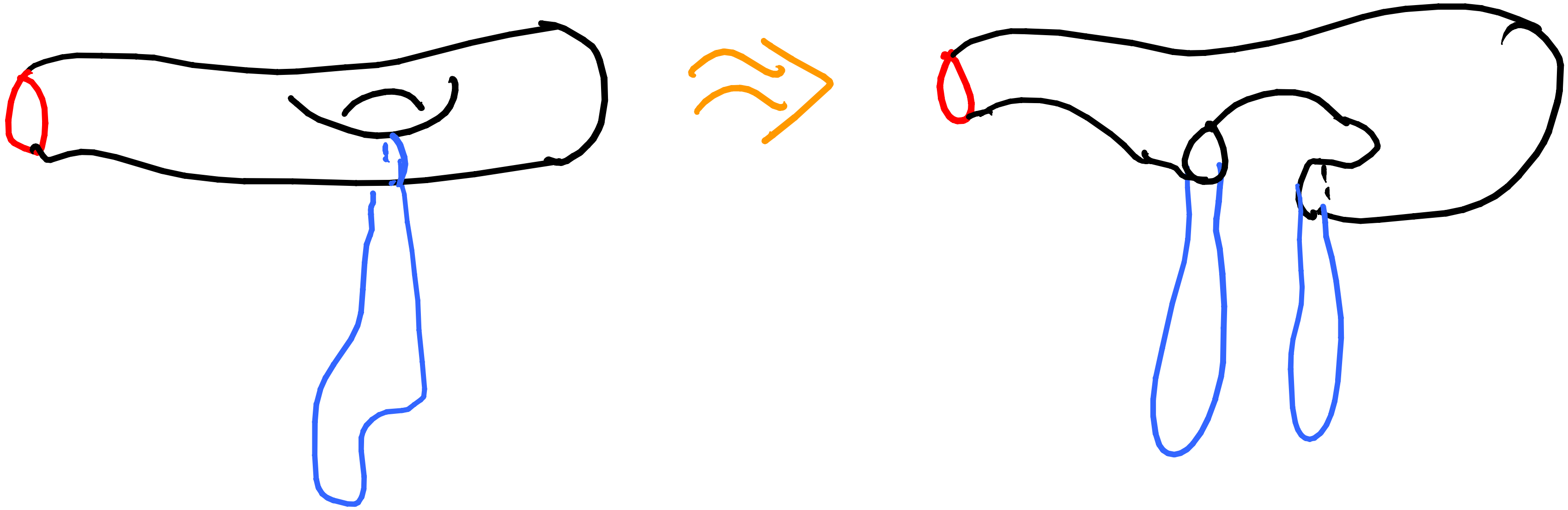
$\forall\, |\omega| = 1 \quad \sigma^*_K(\omega) \equiv \text{signature}\left((1-\omega)V + (1-\bar{\omega})V^T\right)$

Slice $\Longrightarrow$ Algebraically slice $\Longrightarrow$ all $\sigma_\omega(K) = 0$

except ω = root Alex. polyn.

$\therefore$ deep invariants of K are controlled by linking numbers of circles on Seifert surface.

2. If derivative link J is slice link $\Longrightarrow$ K is slice



Theorem (Levine, 69): $K = S^{2n-1} \hookrightarrow S^{2n+1}$ $n > 1$

K alg. slice $\Longrightarrow$ any derivative on any simple Seifert surface is a slice link

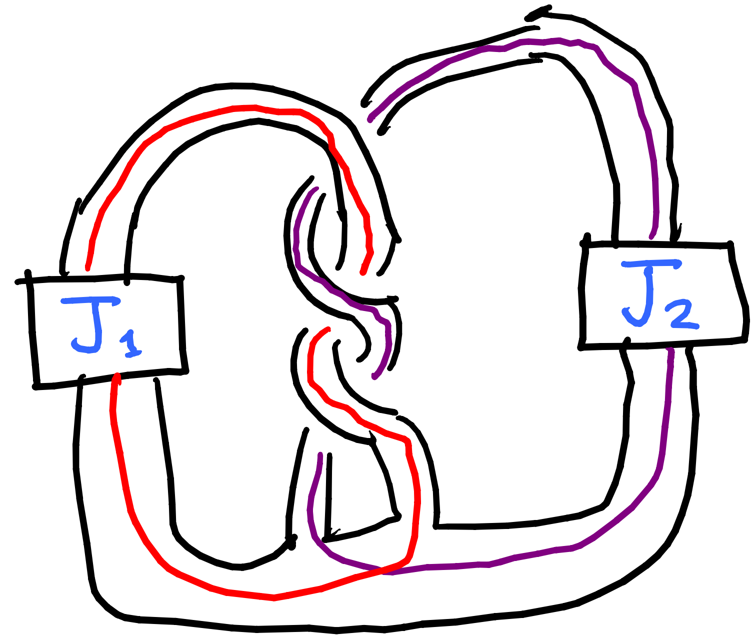
so K is slice.

What about Knots in S^3?

Consider simplest case genus 1 Knot

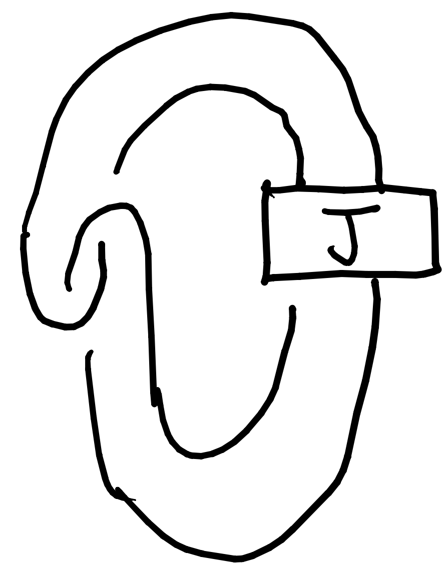
$V = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$ derivative K is a Knot

Example 1: K algebraically slice $\Longrightarrow$ precisely 2 derivatives J_1 and J_2



Example 2: K = Whitehead double of J

Both derivatives have the knot type of J



We know J slice $\Longrightarrow$ K slice

converse???

Conjecture (Kauffman '82): If K is slice knot then, for *any* Seifert surface, one of the derivatives J is slice (hence all signatures of J are 0).

IN THIS TALK:

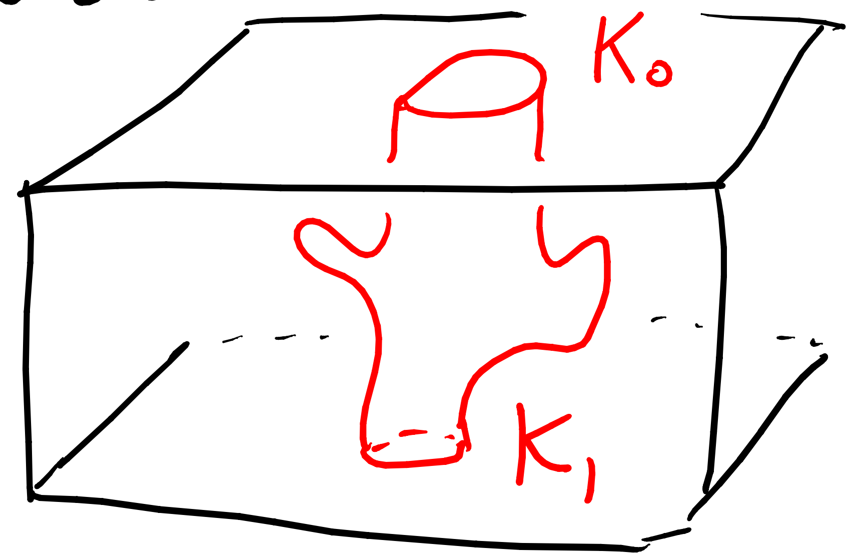
Question 1: K slice $\overset{???}{\Longrightarrow}$ J slice

If not, is J algebraically slice?

If not what CAN we conclude about the associated derivative J?

Filtered version of Question 1

$\mathcal{C}$ = group of concordance classes of knots



Cochran-Orr-Teichner filtration of $\mathcal{C}$ by (rationally n-solvable knots):

$$\subset \mathcal{F}_{2.0} \subset \mathcal{F}_{1.5} \subset \mathcal{F}_{1.0} \subset \mathcal{F}_{.5} \subset \mathcal{F}_{0} \subset \mathcal{C}$$

Casson-Gordon invts. are zero

algebraically slice knots

<u>Easy Lemma</u> (COT): $J \in \mathcal{F}_n \Rightarrow K \in \mathcal{F}_{n+1}$

<u>Question 2</u>: $K \in \mathcal{F}_{n+1} \overset{???}{\Longrightarrow}$ associated derivative $J \in \mathcal{F}_n$?

especially:

$K \in \mathcal{F}_{1.5} \Longleftrightarrow$ associated $J \in \mathcal{F}_{.5}$

"Casson-Gordon invts" of K vanish $\Longleftrightarrow$ signatures of J are zero

History of Kauffman Conjecture:

1. 1976 Casson-Gordon invariants:

$$K \text{ alg. slice } \not\Longrightarrow K \text{ is slice}$$

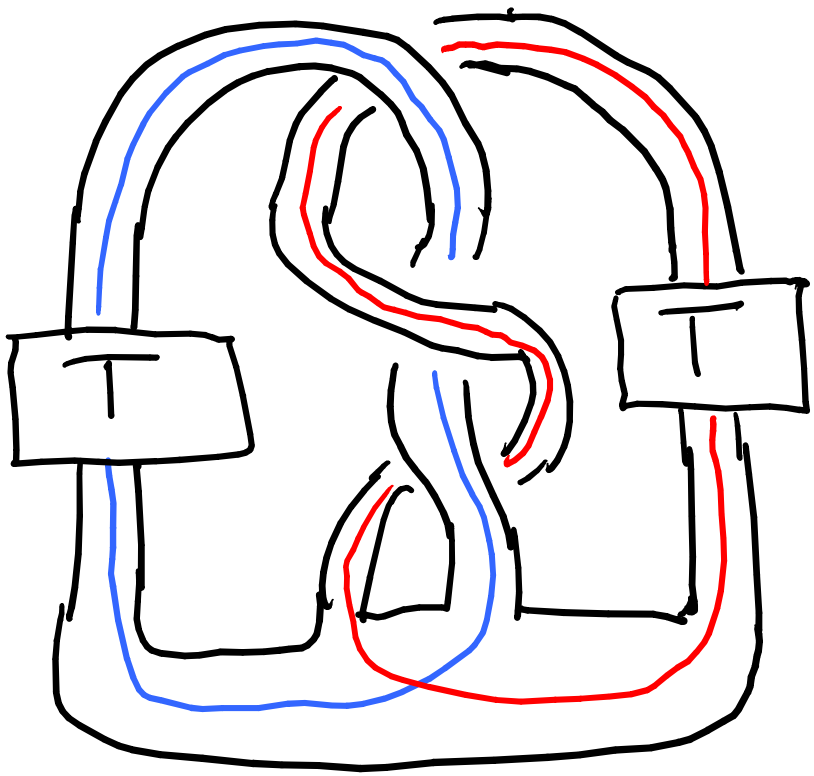
2. 1983 Gilmer: Casson-Gordon invts of K expressed in terms of signatures of J and other invariants of Seifert matrix of J (G-Livingston).

Gilmer-Cooper: $CG(K) =$ sum of signatures of J at p^{th}-root of 1

$\therefore$ K slice (or $K \in \mathcal{F}_{1.5}$) $\Longrightarrow$ certain sums of signatures of J vanish

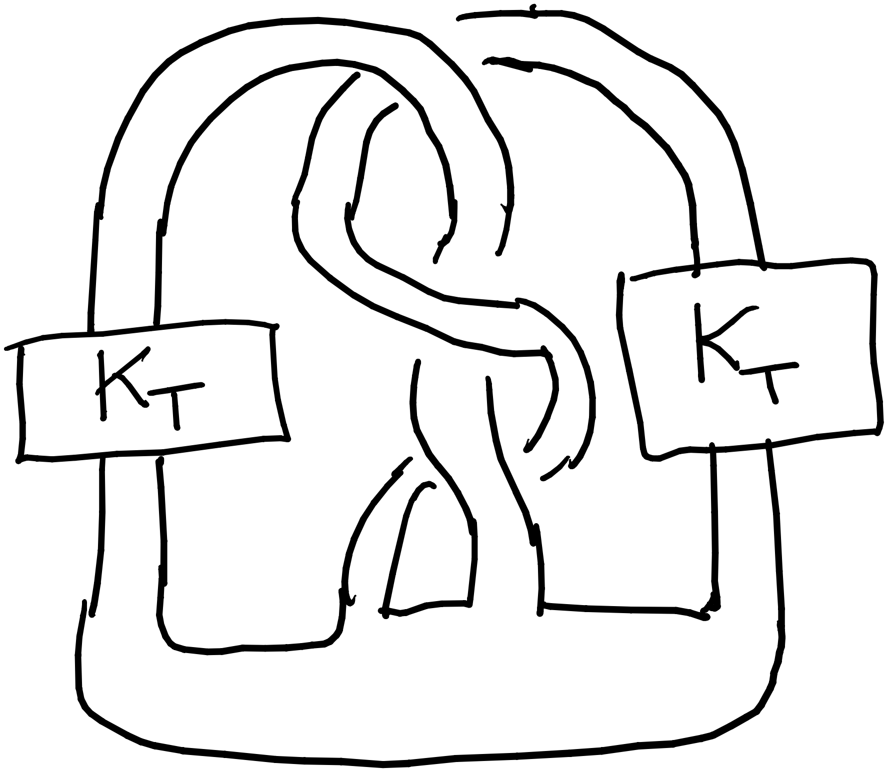
Example

This knot is algebraically slice but many knots, such as $T =$ trefoil have non-zero CG invts since Coopers sum $\sum \sigma_T(\omega) \neq 0$



K_T

3. 2004 COT new higher-order signature invariants
2010 - Harvey-Leidy



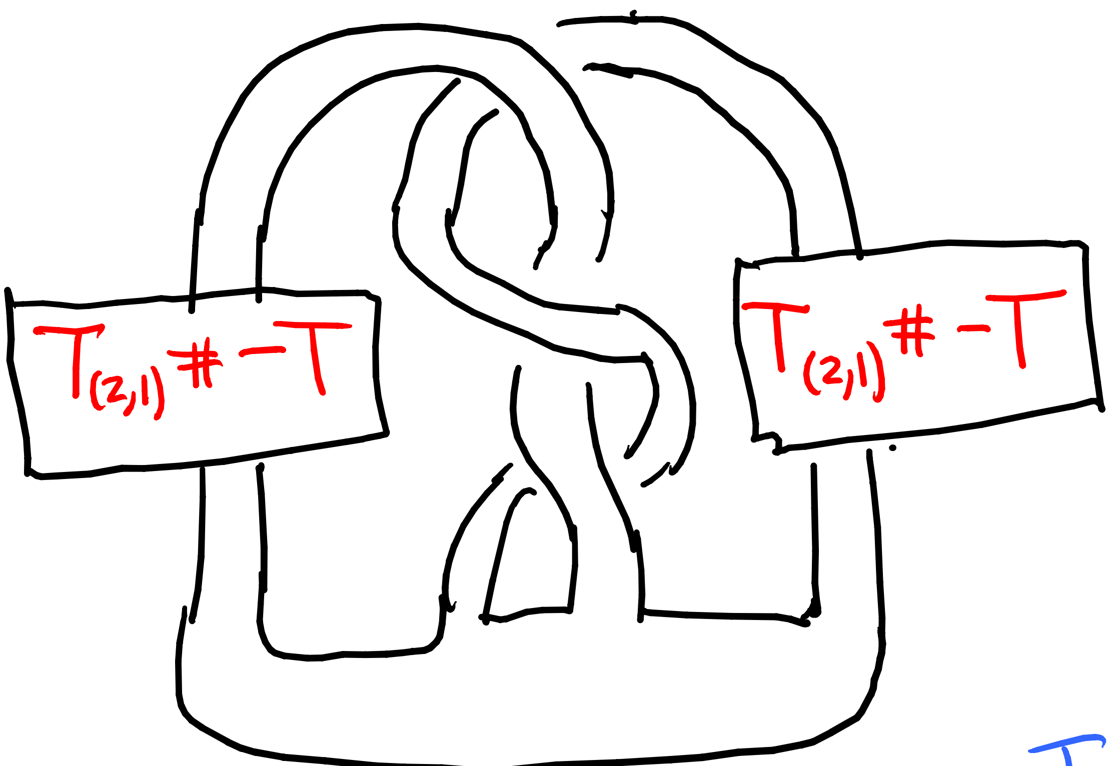
K_T alg. slice so CG of this knot vanish, but not slice

K_{K_T}

$*$ but still expressible as lower-order signatures of K_T

4. Hedden (2007): $\tau(Wh(J))$ is a function of $\tau(J)$.

5. 2013 Gilmer-Livingston Cooper's condition $\sum \sigma_w(J) = 0$ does not imply all signature of J vanish.



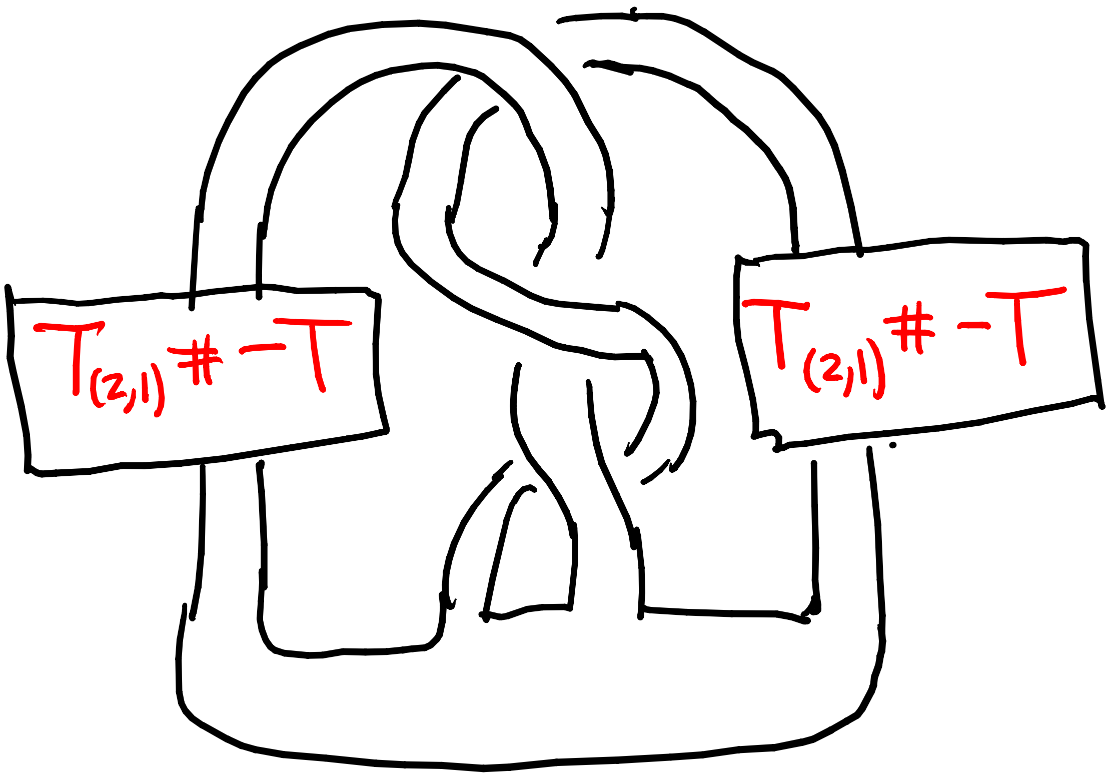
This Knot has vanishing CG invts but $T_{(2,1)} \# -T$ is NOT alg. slice

Is this a slice Knot?

6. <u>Theorem A</u>: (C-Davis 2013) Kauffman's conjecture False. There exist genus one slice knots K for which the associated derivative J has non-zero signatures.

Filtered version is also false:

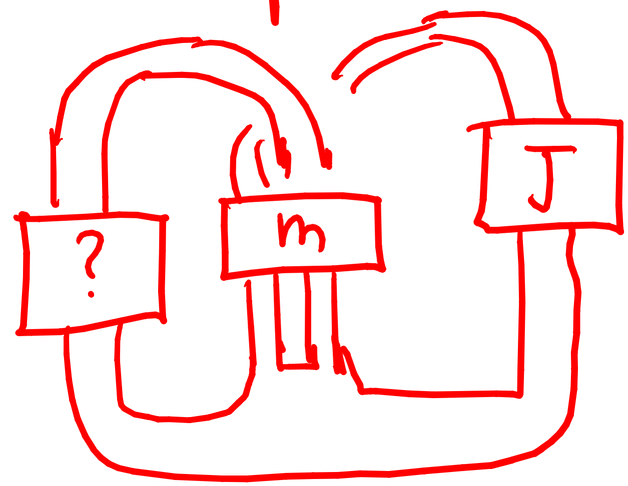
$$K \in F_{1.5} \underset{\text{YES}}{\overset{\not\Longrightarrow}{\Longleftarrow}} \text{derivative } J \in F_{.5}$$



Claim: this knot is in $F_{1.5}$ but $T_{(2,1)} \# -T$ is not algebraically slice

Modification Lemma (C-Davis 2014) Given a genus one knot K with derivative J for any knot T, K is concordant in a homology $S^3 \times [0,1]$ to a genus one knot K' where the associated derivative has algebraic concordance type:

example ↓ ↓



$$[J'] = [J \# T_{(m+1,1)} \# -T_{(m,1)}]$$

cables of T

Corollary 1: Kauffman's Conjecture false

Take J = unknot. Then K is slice so K' is slice with derivative $J' = T_{(m+1,1)} \# -T_{(m,1)}$.

Corollary 2: K genus one as above; derivative J

If $[J_{(c,1)}] = [T_{(m+1,1)} \# -T_{(m,1)}]$ for some $c \geq 1$ and knot T then $K \in \mathcal{F}_{1.5}$.

Corollary 2: Filtered conjecture also false:

$$K' \in \mathcal{F}_{1.5} \Rightarrow \underset{K'=0}{\text{Casson-Gordon}} \;\cancel{\Longrightarrow}\; J' \in F_{.5}$$

(J' alg. slice)

Return to Main Question: If K is slice knot and J is associated derivative link what is true of J?

Main General Theorem: K slice $\Longrightarrow$

"$J = (t_* - id)T = t_*(T) - T$"

for some link 'T' in rational homology S^3.

What does this mean!!!

recall for genus one knots K we saw example

$$J = T_{(2,1)} \# -T$$

Let M_J = 3-manifold = 0-framed surgery on J

Recall there is characterization of J being slice in terms of M_J:

<u>Well-Known</u>: J is slice in homology B^4

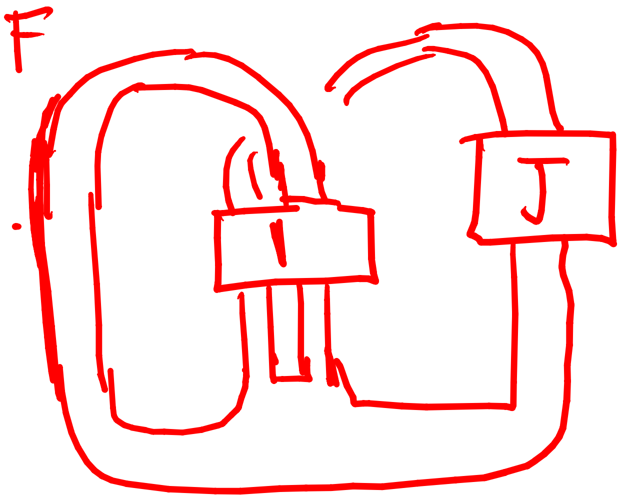
$\iff M_J = \partial W$ s.t. $H_2(W) = 0$ and $H_1(M_J) \cong H_1(W)$.

Main General Theorem If K is slice and J is associated derivative link then $\exists$ 4-manifold W^4 with $H_2(W)/H_2(\partial W) = 0$

and $M_J \amalg M_T \amalg -M_T = \partial W$ for some link T in some rational homology 3-sphere, and

- extra π_1, H_1 conditions

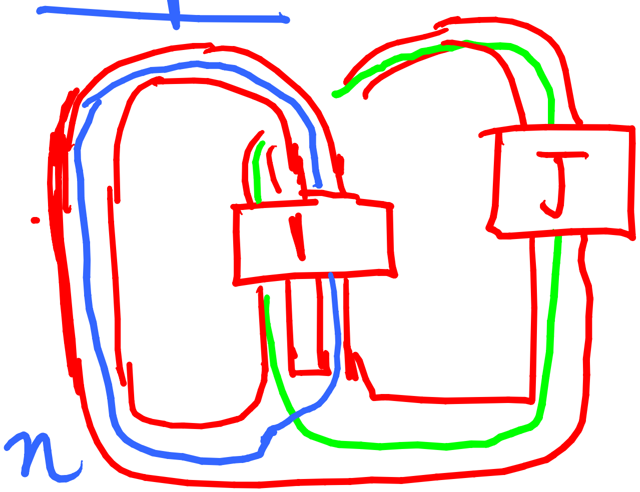
Sketch of Proof of Modification Lemma:



K

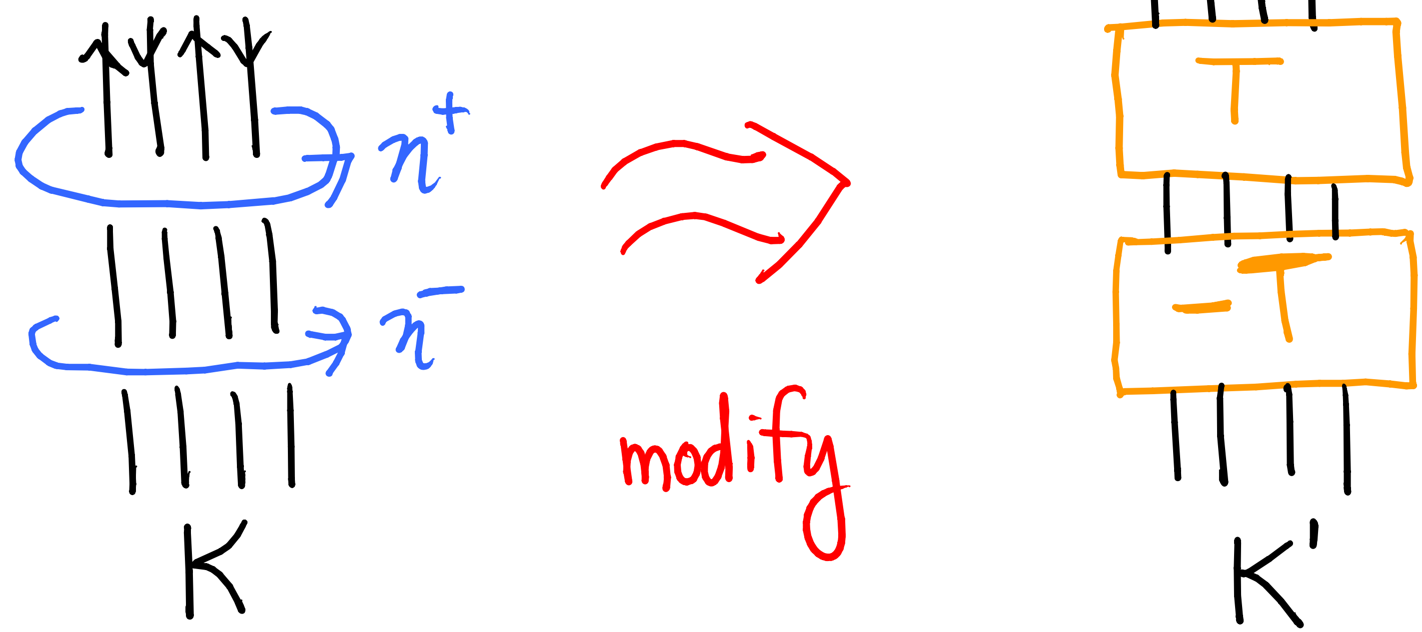
Modify (K, F, J) by concordance to (K', F', J') where $[J'] = [J \# T_{(2,1)} \# -T]$ for any prescribed knot T.

Step 1: choose dual curve η to J $\quad \eta \cdot J = 1$

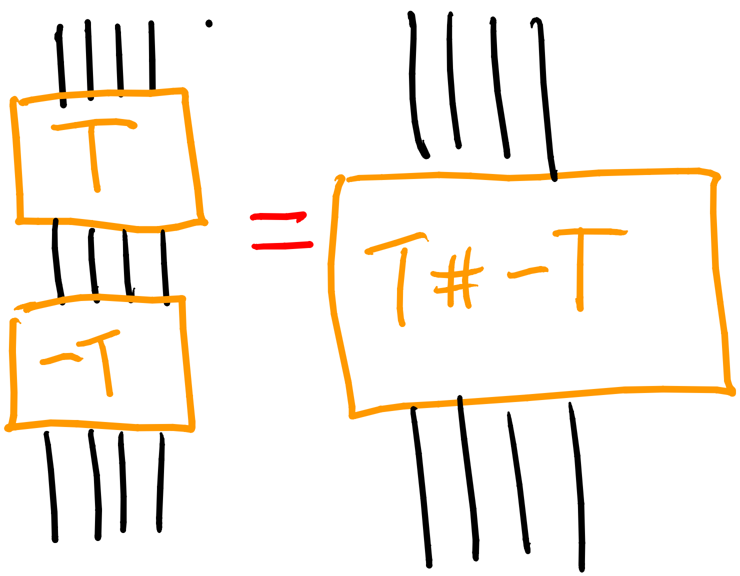


Form 2 push-offs η^+, η^- of η cobound annulus

Schematic Picture

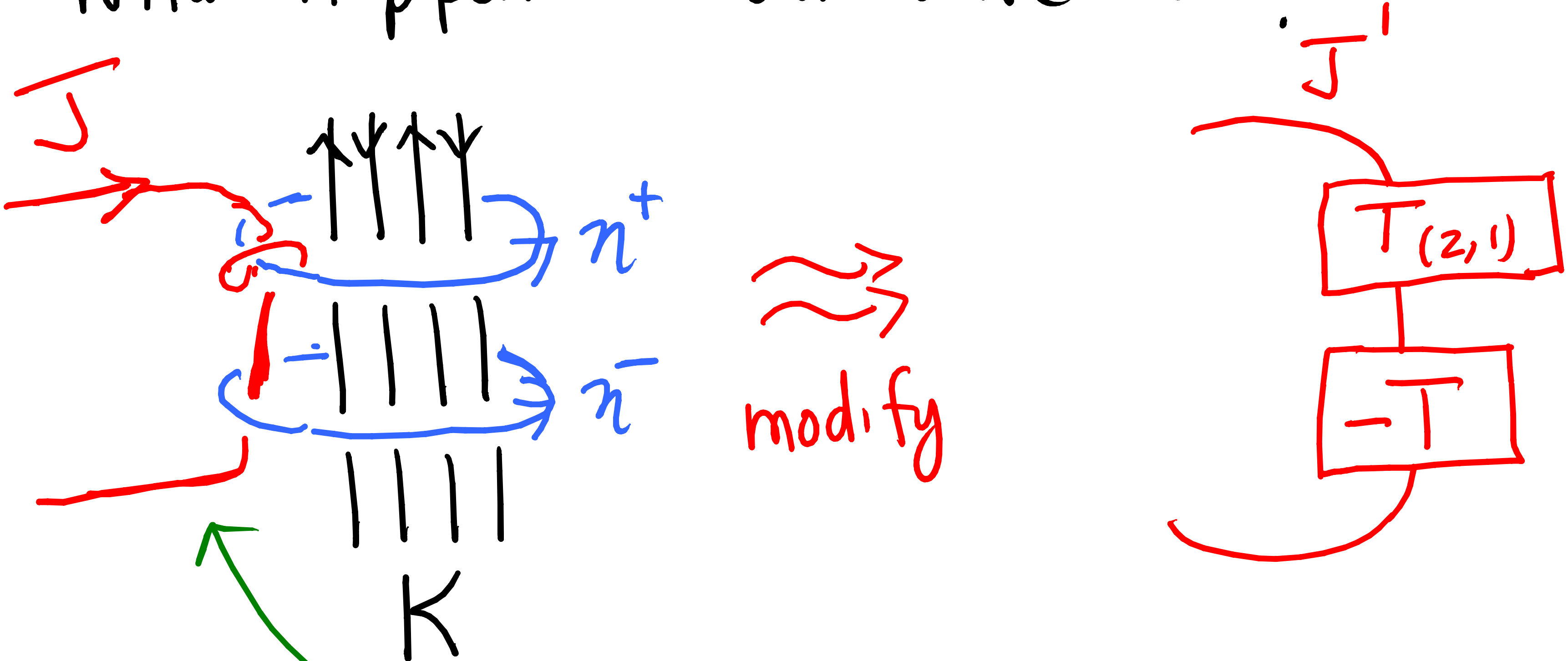


K' is concordant to K since



and T#-T is a slice knot for any T

What happens to derivative J ?



Since $\eta \cdot J = 1$, $lk(J, \eta^+) \neq lk(J, \eta^-)$

* Note: $\eta^+ = t_*(\eta^-)$ in Alexander Module K !!

so $\eta^+ - \eta^- = (t_* - 1)\eta_-$

<u>Corollary 3</u>: K genus one as above; derivative J

If $[J] = [T_{(2,1)} \# -T]$ for some knot T then $K \in \mathcal{F}_{1.5}$.

<u>Step 1</u>: Apply Modification Lemma to get concordant knot K' such that $[J'] = 0$, i.e. J' is algebraically slice.

<u>Step 2</u>: Apply COT, $J' \in \mathcal{F}_{0.5} \Rightarrow K \in \mathcal{F}_{1.5}$

Sketch of proof of Main General Theorem:

Assuming K is slice find 4-manifold $W \longrightarrow$

$H_2(W)/H_2(\partial W) = 0$

Step 1: Recall K slice $\Longrightarrow M_K = \partial Y^4$ with $H_2(Y) = 0$

Easy $Y = B^4 -$ slice disk

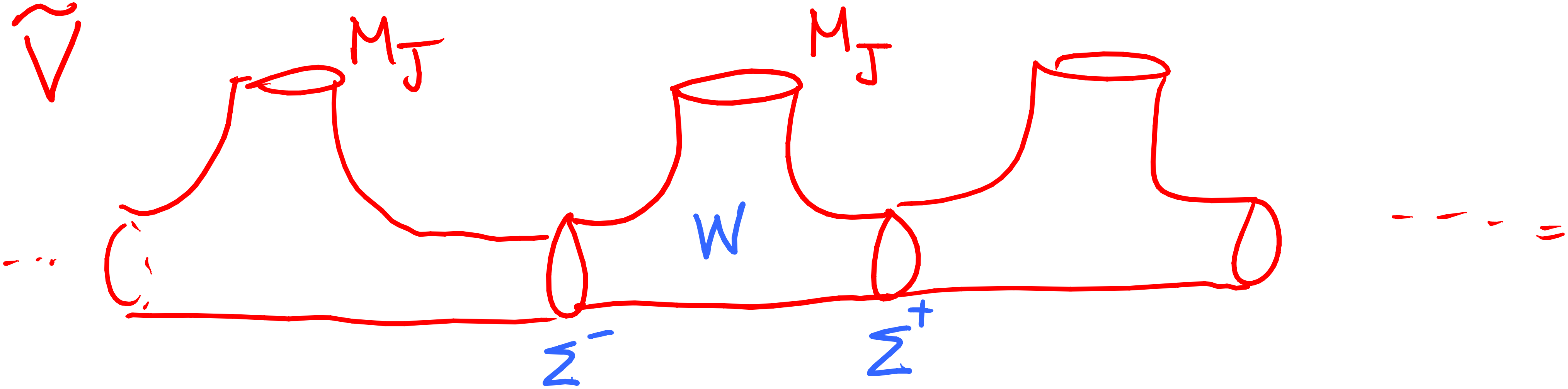
Step 2: There is a <u>fundamental cobordism</u> E relating 0-surgery on K and 0-surgery on its derivative.

Step 3: Let $V = E \cup_{M_K} Y$

note $H_1(V) \cong H_1(B^4 - \Delta) \cong \mathbb{Z}$

take ∞-cyclic cover of V by cutting open along 3-manifold Σ

$\tilde{V}$



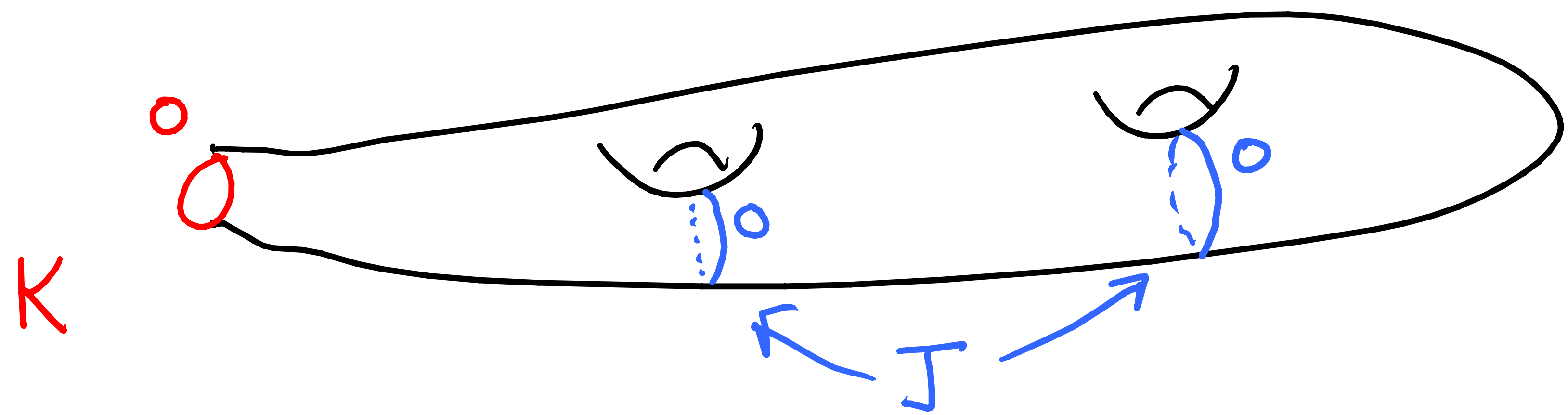
Let W^4 = fundamental domain = $V - N(\Sigma)$.

FACT: Any 3-manifold Σ is zero framed surgery M_T on some link T in a rational homology 3-sphere.

Note: Σ^+ really is $t_*(\Sigma)$!!!!

Construction of fundamental cobordism between M_K and M_J J = derivative of K.

1. Add 0-framed 2-handles to $M_K \times [0,1]$ along components of J, call it D



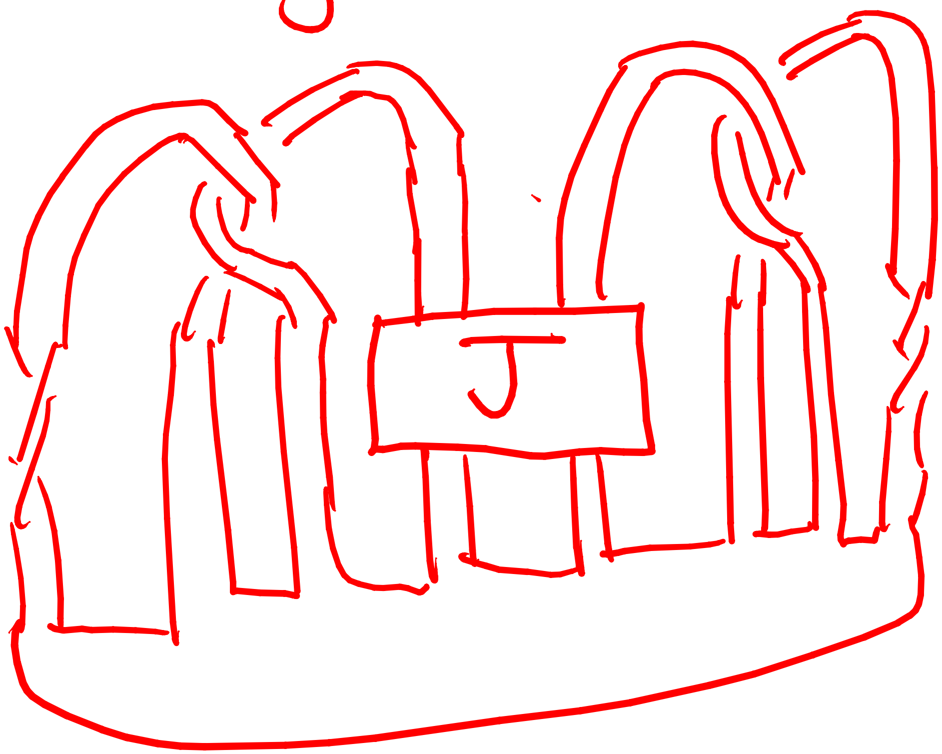
2. K is unknotted so $\partial D = M_K \sqcup M_J \# S^1 \times S^2$

3. Let $E = D \cup$ 3-handle

See talk of Christopher Davis for another application of Main Theorem

Different view of Main Theorem:

Any genus g knot K is concordant to a knot obtained from a ribbon knot R by infection on a g-component string link J:



$= R(J)$

We have an "exact sequence"

$$\mathcal{SL}^{\infty} \xrightarrow{\text{"}t_* - id\text{"}} \mathcal{SL}^{g} \xrightarrow{R} \mathcal{C}$$

$$J \longrightarrow R(J)$$

concordance classes of g-component string links with zero linking numbers