4-Dimensional Equivalence Relations on Knots

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* slides will be soon on my web page along with suggested references added to final page of talk.

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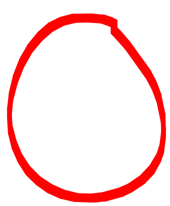
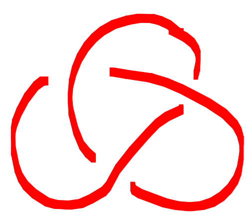
Def: A knot is a smooth embedding

$$f : S^1 \longrightarrow \mathbb{R}^3$$

i.e. take a rope, tie it up and attach the ends



unknot



trefoil

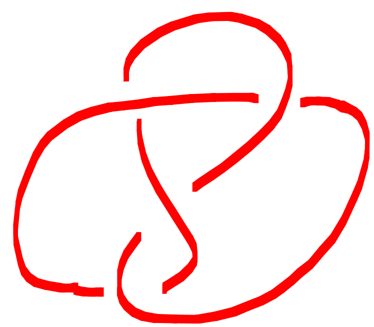
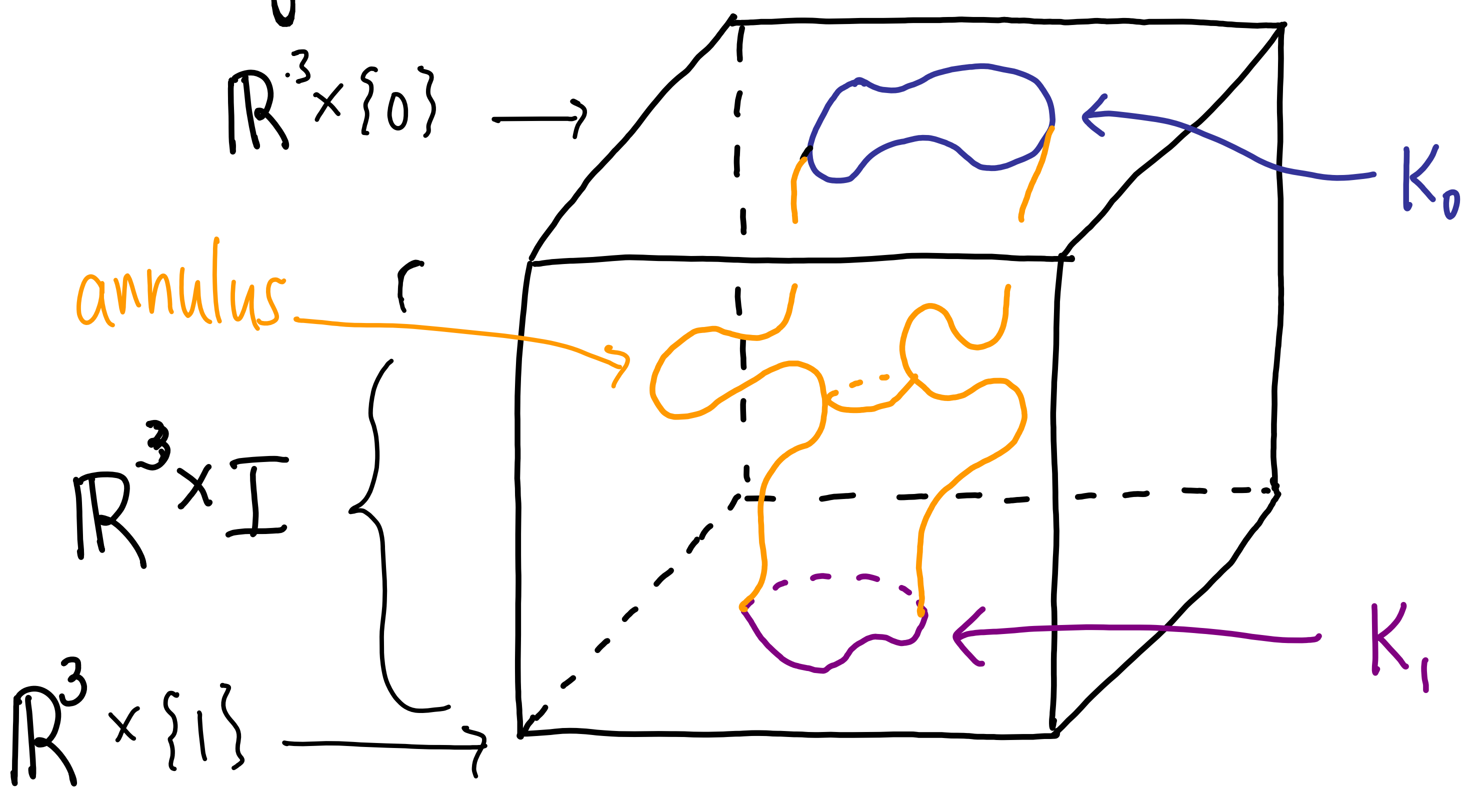


figure-eight

Def: Two knots are equivalent if one can be deformed into the other through embeddings in $\mathbb{R}^3$

Change this to $\mathbb{R}^3 \times [0, \infty)$ to get different 4-DIM equivalence relation

Def: Knots K_0 and K_1 are **Concordant** if $K_0 \times \{0\}$ and $K_1 \times \{1\}$ cobound a smoothly embedded annulus in $\mathbb{R}^3 \times [0,1]$

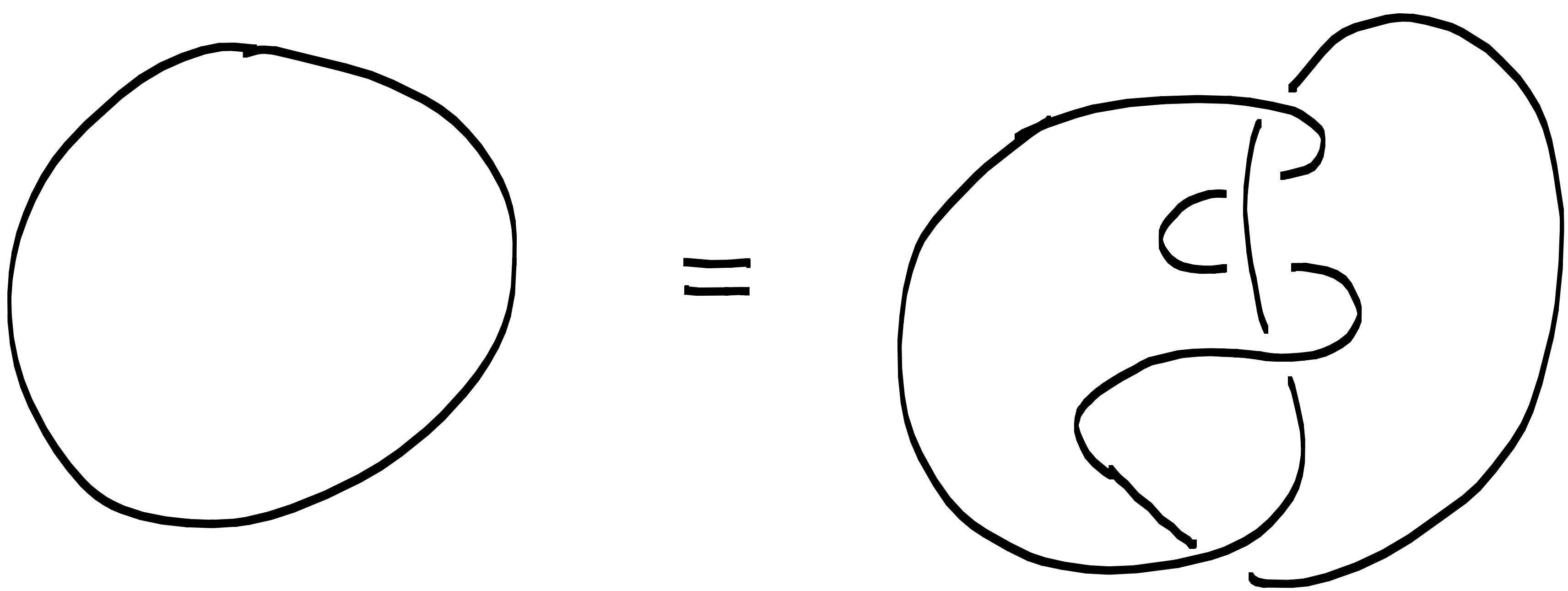


Question: Which Knots are concordant to trivial KNOT?

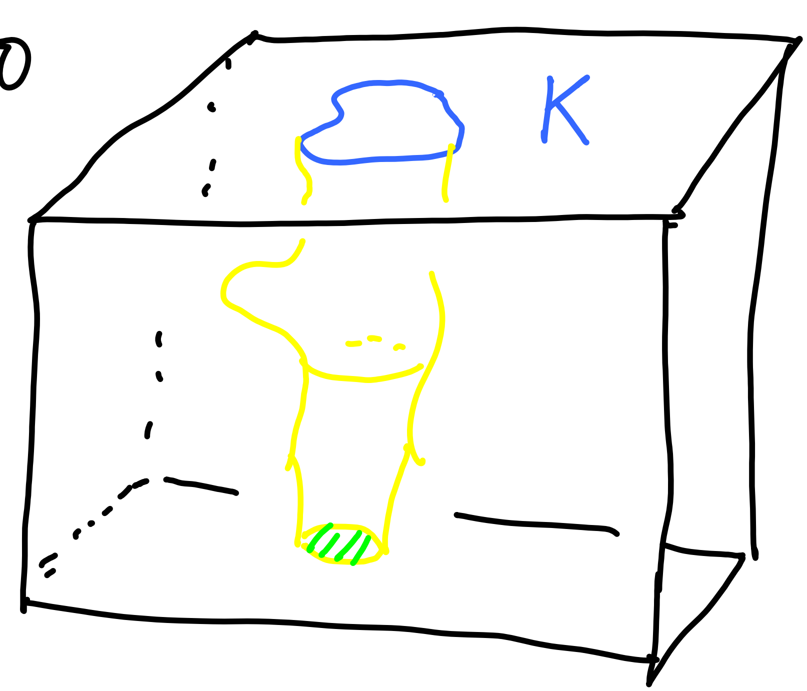
In 3-dimensions a Knot is equivalent to the trivial Knot

$\Updownarrow$

it bounds a disk embedded in $\mathbb{R}^3$



If K is concordant to trivial knot, then it bounds a D^2 in $\mathbb{R}^3 \times [0,1]$.

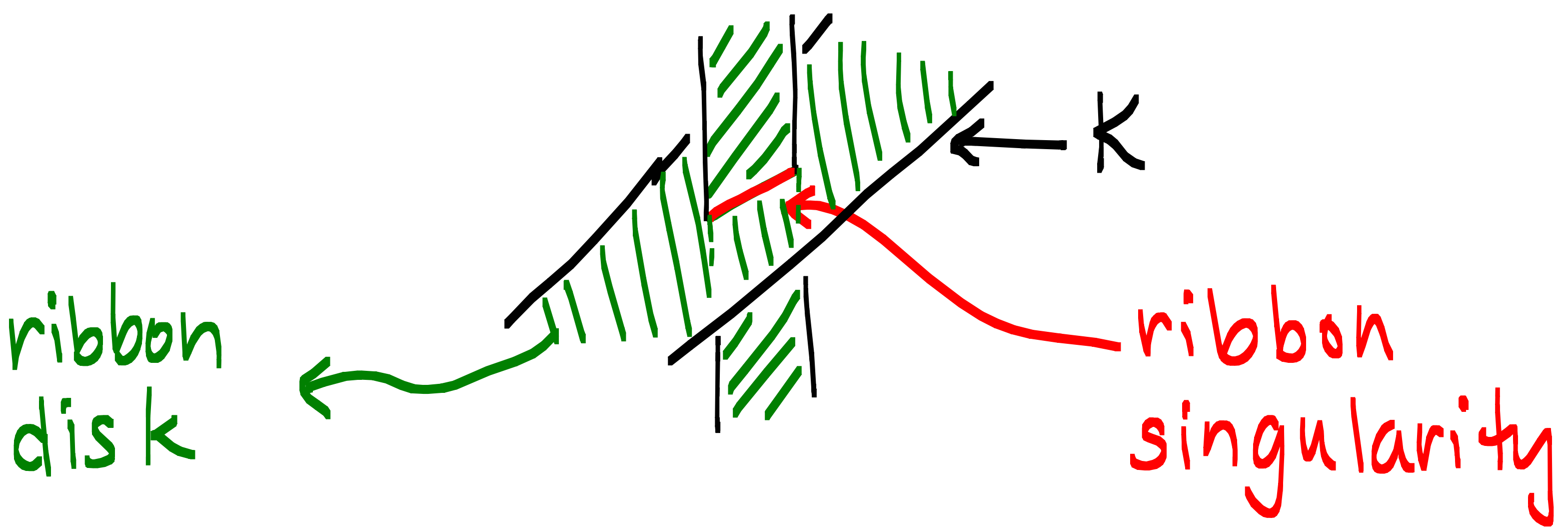


Def: A knot $K \subset \mathbb{R}^3$ is **slice** if it is the boundary of a smoothly embedded disk in $\mathbb{R}^4_+$.

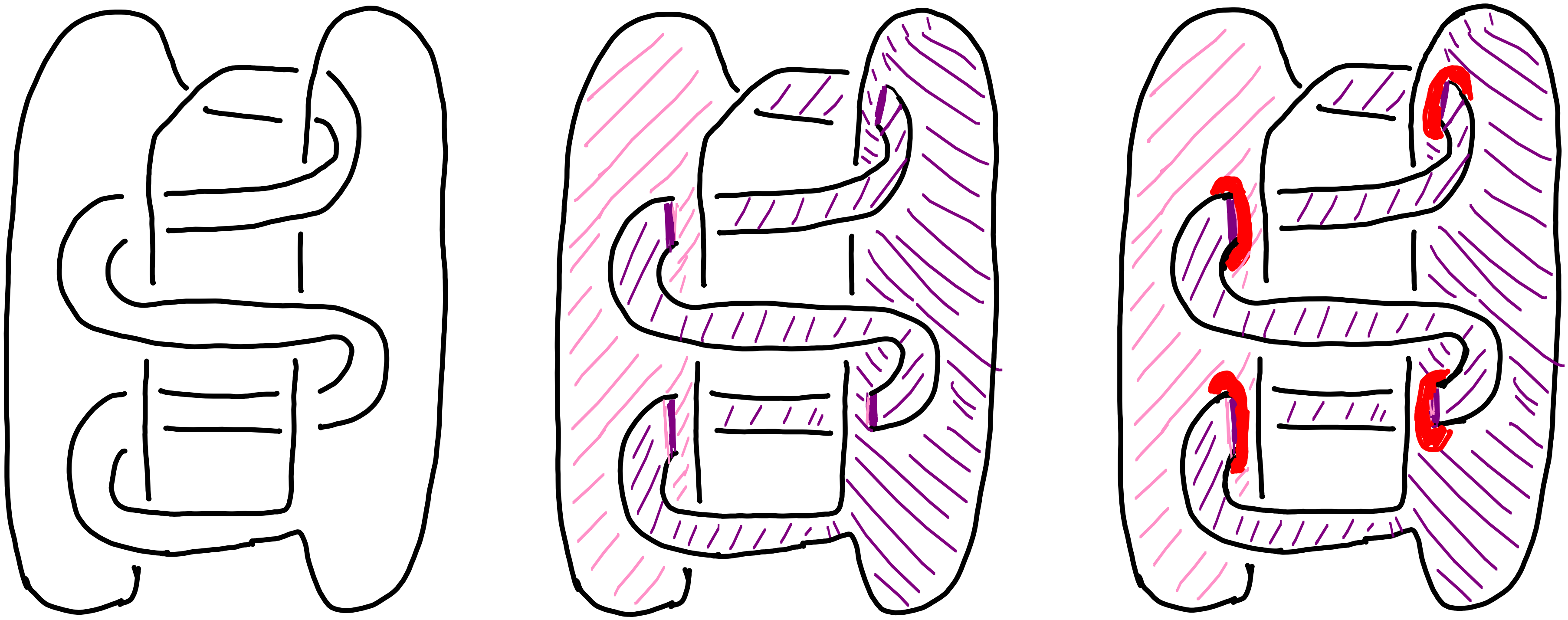
Such disk called a **slice disk for K**.

Examples of slice knots:

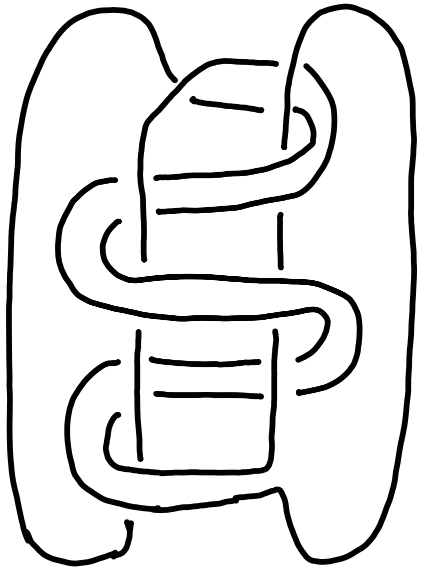
Def: K is ribbon if it bounds an immersed disk in $\mathbb{R}^3$ with only ribbon singularities.



A ribbon knot is slice



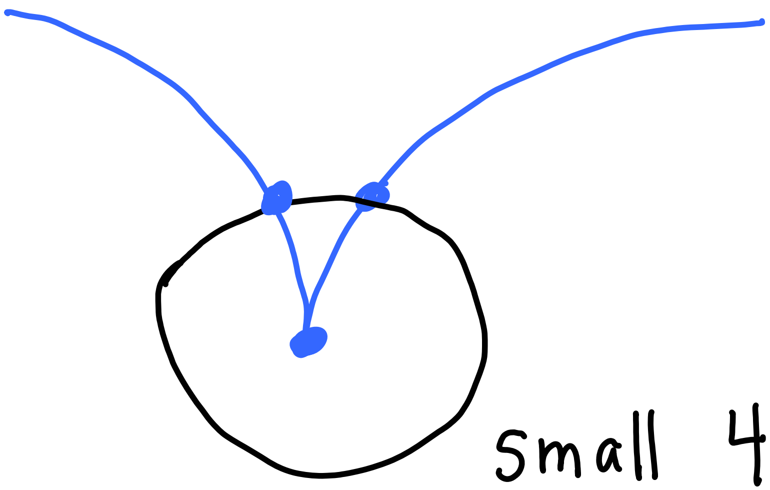
Pf: To obtain a disk <u>embedded</u> in $\mathbb{R}^4_+$, push the interior of red disks into interior of $\mathbb{R}^4_+$.

So  is slice.

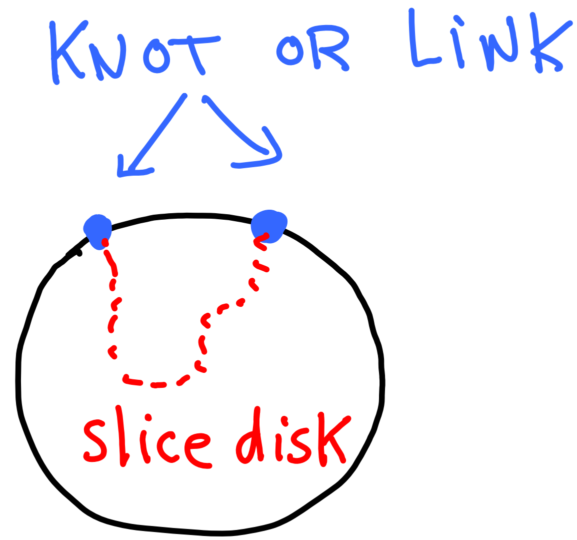
Old Conjecture: A knot is (smoothly) slice $\Leftrightarrow$ it is a ribbon Knot

Why do we care?

1960 Milnor, Fox, Kervaire: isolated singularities of surfaces in 4-manifolds



small 4-ball



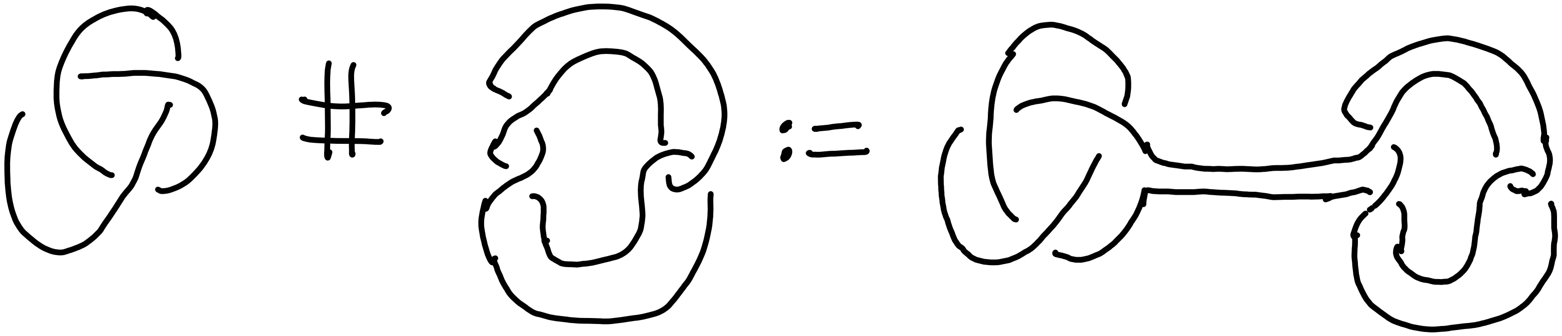
$\partial B^4 = S^3$

if "link of singularity" is slice, singularity can be resolved.

$$\frac{\text{Knots}}{\text{isotopy}} \longleftrightarrow \text{classification of 3-mfds}$$

$$\frac{\text{Knots}}{\text{concordance}} \longleftrightarrow \text{classification of 4-manifolds}$$

binary operation on knots: connected sum



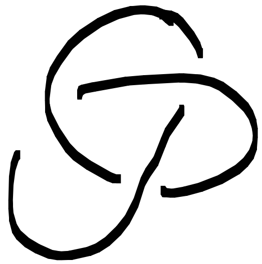
K_1 $\qquad$ K_2 $\qquad$ $K_1 \# K_2$

<u>Def</u>: The knot concordance group $\mathcal{C}$ is $\dfrac{\{\text{knots}\}}{\text{concordance}}$ with connected-sum as operation.

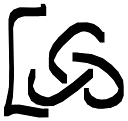
$0 = \{\text{slice knots}\}$ $\qquad$ inverse = mirror image

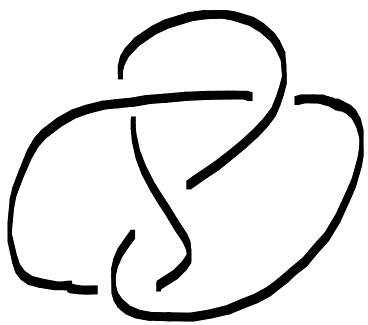
What is $\mathcal{C}$?

1. Is it finitely-generated?
2. Are there elements of finite order?
3. Are there elements that are infinitely divisible? Is $\mathbb{Q} \subset \mathcal{C}$?
4. Is there a complete set of invariants that determine if a knot is a slice knot ($[K]=0$ in $\mathcal{C}$)?

Ex:  is not slice

$\Rightarrow$ $\mathcal{C}$ is non-trivial

In fact, [] has infinite order in $\mathcal{C}$.

Ex: $K =$  $-$Amphichiral knot

$$2[K] = [K \# K] = [K \# -K] = [O]$$

$\therefore$ Any $-$amphichiral knot has order 2 or is itself slice.

What is Known?

There is a filtration

(Cochran - Orr - Teichner 1998)

$$0 \subset \cdots \quad F_{n+1} \subset F_n \subset \cdots \subseteq F_2 \subset F_1 \subset F_0 \subset \mathcal{C}$$

$$\mathcal{C}/F_1 \cong \mathbb{Z}_1^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$$ 1960's Levine, Stoltzfus

$$\mathbb{Z}_2^\infty \oplus \mathbb{Z}^\infty \subseteq F_1/F_2$$ 1970's Casson-Gordon, Jiang, Livingston

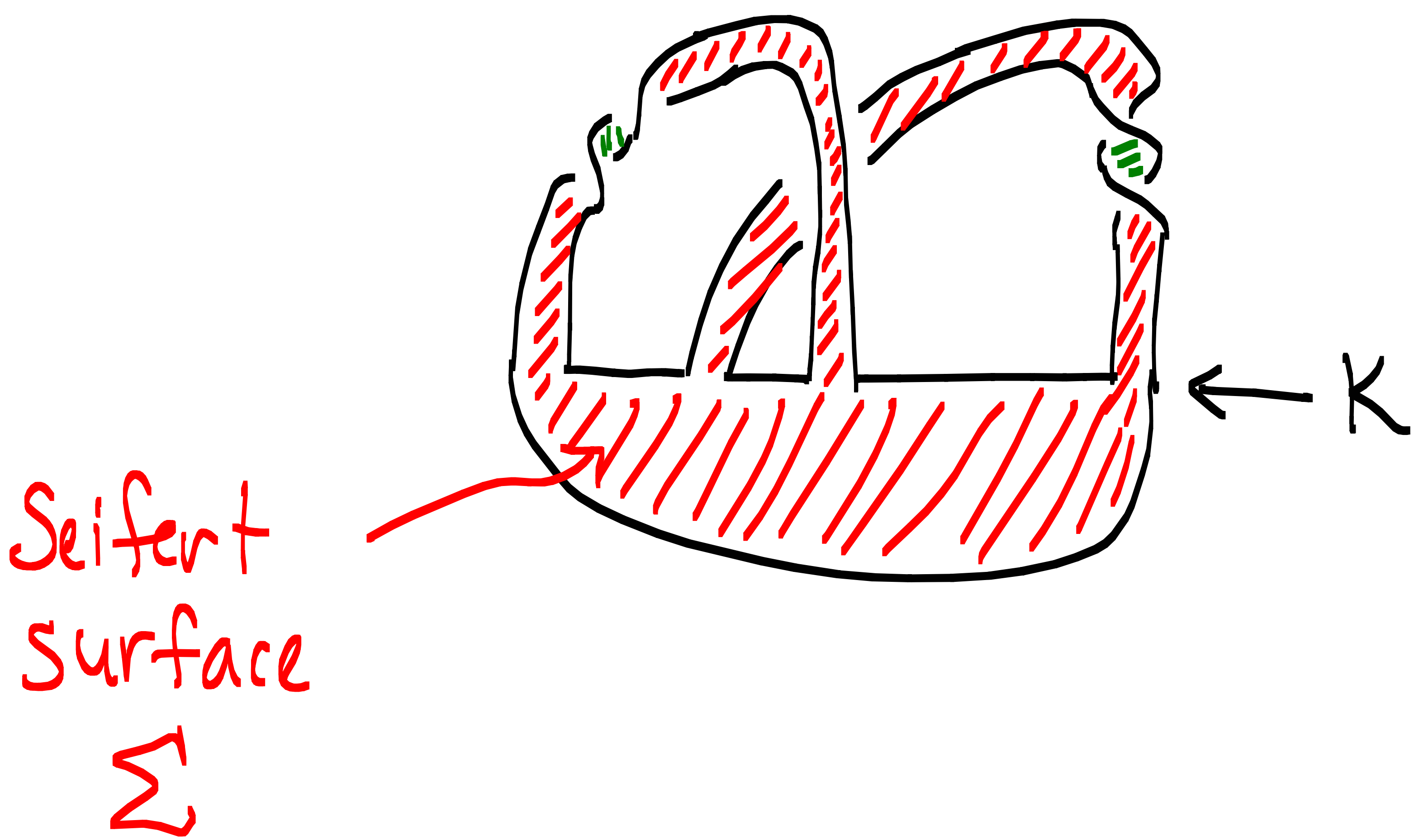
$$\mathbb{Z}_2^\infty \oplus \mathbb{Z}^\infty \subseteq F_n/F_{n+1}$$ 2000's Harvey, Leidy, Cochran

Tools:

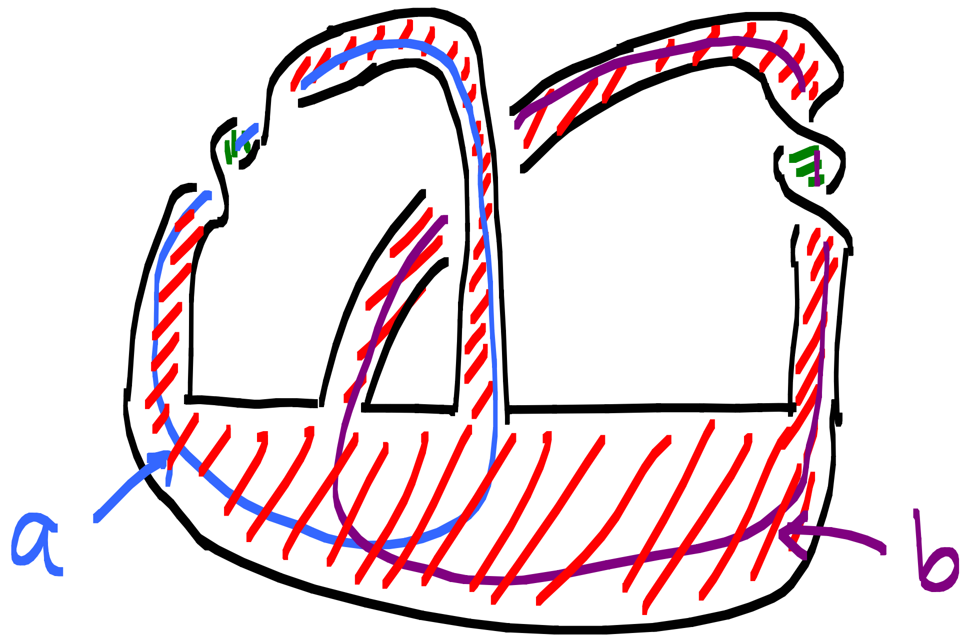
- signatures, invariants of bilinear forms
- covering spaces, homological algebra, group theory
- Heegard Floer Homology
- Khovanov Homology
- NOT GEOMETRY

Next: a specific set of invariants defined by Tristram used to show a knot is not slice

Def: A Seifert surface Σ for K is a 2-sided surface embedded in S^3 with $\partial\Sigma = K$.



From a Seifert surface ⇝ Seifert matrix



$$V = \begin{pmatrix} lk(a,a^+) & lk(a,b^+) \\ lk(b,a^+) & lk(b,b^+) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

For $w \in \mathbb{C}$, $\|w\| = 1$

$(1-w)V + (1-\bar{w})V^T$ is a Hermitian matrix

Def: $\sigma_w(K) := \text{signature of} \left((1-w)V + (1-\bar{w})V^T\right)$
$\in \mathbb{Z}$.

$$\boxed{\begin{array}{l}\text{Thm: If } K \text{ is slice and } w = (p^k)^{th} \text{ root} \\ \text{of unity} \Rightarrow \sigma_w(K) = 0.\end{array}}$$

$\rightsquigarrow \quad \oplus \sigma_w : \mathcal{C} \longrightarrow \mathbb{Z}^{\infty}$

Conjecture: $\mathcal{C}$ is a fractal set, i.e. $\mathcal{C}$ embeds in itself in ∞-many ways (via satellite operations)

Open:

1. are there elt's of finite order except 2
2. are there any elt's of order 2 except explained by amphichiral?
3. any ∞-divisible elt's?