Rice University
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Colloquium
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Knot Concordance Group
Filtrations of the
A knot is a smooth embedding

\[ S^1 \to \mathbb{R}^3 \]

\[ \gamma : \mathbb{R} \to \mathbb{R}^3 \]

Ex: Figure-8 knot

Ex: Right handed trefoil

Ex: Left handed trefoil

\[ (z, w) \in \mathbb{C} \times \mathbb{C}, |z|^2 + |w|^2 = 1 \]

\[ \mathbb{S}^2 \to \mathbb{R}^3 \]
Consider the complex curve \( C \) defined by
\[
\mathbb{Z}_3 - M_3 = 0.
\]
The singularity at \((z, w) = (0, 0)\).

The link of the singularity is
\[
\{ z \in \mathbb{C} | |z| + |w| = 3 \}.
\]
This is a 1-dimensional (real) curve in \( S^3 \) (i.e., a knot or link).

Several components

\( A \) is \( S^3 \) = \( S^3 \).
\[ L = \{ (R, \theta) \in \mathbb{R}^2 \times \mathbb{R} \mid R \neq 0, \theta \neq \frac{2}{3} \pi + \frac{k}{3} \pi, \text{ for } k \in \mathbb{Z} \} \]
Through itself, the other in $S^3$ without passing the other in $S^3$ equivalent if you can deform one into the other up to isotopy: two knots are equivalent if you can deform one into the other up to isotopy.

In knot theory, one usually studies...
$\circ = \bigcirc \neq \bigcirc$
Examples of Distinct Knots up to Isotopy

Left-handed Right-handed
Trefoil Trefoil
Figure 8 Trivial knot
The trivial knot $O$ is the only knot that bounds an embedded disk in $S^3$. 

\[ \text{Diagram: A simple circle} \]
There is a binary operation on knots:

\[ K \# K = \bigcirc \quad = \bigcirc \# \bigcirc \]

\( K \) and \( K' \), and \( K_1 \) and \( K_2 \).

Connected sum of knots:

\[ K \# K' = K_1 \# K_2 \]
new equivalence relation called concordance.

To get a group structure, define a

\[ \# K = 0 \]

Thus \[ \# (\{\text{knots}\}, \#) \text{ forms a} \]

Exercise: there is no knot \( K \) such that

does not have inverses.

However \( K \) is not a group since it

Moreover with unity = 0.
$B_{\epsilon} = \{ (z', m)(z, m) | |z'|^2 + |m|^2 = \epsilon^2 \}$

$\Sigma = \partial B_{\epsilon} = \{ (z, m) | |z|^2 + |m|^2 = \epsilon^2 \}$

In $B_{\epsilon} = \mathbb{H}_{\text{dim. ball}}$, $D$ is a 2-dimensional disk (smoothly) embedded.

DEF: A knot $K$ is $\Sigma$-slice if $\Sigma = \partial D$ where $\partial \Sigma$ is trivial.

We will think of "slice" knots as "trivial".
Singularity is neverSilicel

However, if turns out that the link of a
can replace singularity with a smooth disk.

• If the link of a singularity is Silicel, remove a plane curve singularity.

Knot being Silicel to understand when one could

Fox-Milnor first studied the notion of a
To obtain embedded in $B$, push interior of red disc. Into interior of $B$.

$8^b = \pi(\text{im} \text{merged})$

Example: Any ribbon knot is slice.
The 9_{46} knot is slice
Proof: "Spin" $K$ through $\mathbb{R}^4$. $K$ = mirror image of $K = reverse all crossings $]$

If $K$ is any knot then $\#K$ is slice.
Since \( Af(h') \neq 0 \),

\[
\overline{h' \# h'} = \overline{h'} \# \overline{h'} \text{ is \textit{Silico}}.
\]

Claim: \( h' \) is not \textit{Silico} but \( h' = h \).

So

(Figure-8)

(exercise) \( h' = h \):

\[
\text{Exercise: } h' = h.
\]
If $K$ is slice then $\rho_0(K) = 0$.

For we define

\[ \sigma_0(K) := \sum \omega(K)(1-w)(1-w^2)v^t \]

\[ \rho_0(K) := \sum \omega(K) dw \]

where

\[ k = \omega \text{(surface)} \]

Levine-Tristram signature: Sliceness Obstructions

\[ V = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \]

\[ \text{linking matrix} \]

\[ \beta \text{link}(a,b^+) \]
So triangle is not slice.

\[ p_0(\text{triangle}) = -\frac{4}{3} \neq 0 \]

\[ \delta_0(\text{triangle}) = -2 \]

\[ \delta_\ell(\text{triangle}) = 0 \]

\[ \ell_\delta(\text{triangle}) = 0 \]

\[ w = \frac{2}{1 + \sqrt{3}} \]

\[ \varphi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

\[ \omega = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \]

EX: Triangle is not slice.
Def: Knots $K_0$ and $K_1$ are concordant if $K_0 \times \mathbb{I}$ and $K_1 \times \mathbb{I}$ cobound a smoothly embedded annulus in $S^3 \times \mathbb{I}$. 

In $S^3 \times \mathbb{I}$ the knot $K_1 \times \mathbb{I}$ is concordant to the trivial knot.
\[ 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[ \text{K is slice } \iff [K] = 0 \]

\[ [\begin{pmatrix} \ast & g \\ \ast & 0 \end{pmatrix}] = [g] + [0] \]

---

\[ \text{Connected sum of knots.} \]

\[ \text{G is an abelian group under the operation} \]

\[ \text{\texttt{Def.}} \ \text{C} = \{ \text{knots in } S^3 / \text{concordance} \} \]
is not slice but 0 \# is slice.

[0] is a torsion in \( \mathcal{E} \) since

\[ \text{Note:} \]

\[ [x] = [0] \]

\[ \text{-} \]

\text{K is slice.}

K is of the inverse of [K] is [K] slice.
$\ker \frac{\text{def}}{\text{det}} \subseteq \text{ Algebraically Slicely Knots}$

$\text{Group} \quad \begin{array}{c}
\mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}^4 \\
\mathbb{Z}^2 \end{array}$

$\text{Algebraic Canonical} \leftarrow \text{Slicely Knots}$

To define epimorphism (including signatures and Art invariant)

Including signatures and Art invariant

Obtained from Seifert matrix

In late 60's Levine used invariants

In Milnor–Turaev used signatures

$67$, $67$
Invariants to show $\eta_{46}(K)$ is not Slicee.

Signatures, Gilmer used Casson-Gordon

For $K$ with certain non-vanishing

slice algebraically called

\[
\text{slice} = \eta_{46}(K)
\]

Slice in band

tie K

Ex: \[ \eta_{46} = 0 \]
K into knot
the string

\[ I + K = \text{if } k \]
Gordon invariants.

0.5 c Knots with Vanishing Casson-

osity.

0.5 = Algebraically Slice Knots

0.5 = Art Invariant Zero Knots

0 = Slice Knots, \{ c \cdots c, c \cdots c, c \cdots c \} = C

The (n)-Solvable Filtration of G (new)

In 1997, Cochran- Orr- Teichner defined
\[(A, B) = \{ a \mid \text{a is } A \} \text{ and } (B, A)\]

where

\[
g^{(n)} = \begin{cases} g, & g \in G_n \text{ for } n \geq 0, \\ (\varepsilon, \varepsilon), & n = 0. \end{cases}
\]

If \( G = \text{group} \Leftarrow \) derived series defined as

\[
\text{\"cube of 3-neighbor\" (K)}.
\]

\[
(S_3 \text{ -neighbor(K)} \cup \text{solid torus}) = \text{K} = \text{0 - surgery on K (closed 3-manifold)}
\]

If \( K \) is a knot,
If \( \tilde{\pi}_1'(\mathcal{T}) \subset \mathcal{T} \), \( \mathcal{T} \) as well then \( K \) is solvable.

(2) \( H_2(W) \) has a basis \( \{\xi_i\}_{i=1}^g \) of embedded surfaces (with trivial normal bundle) all disjoint.

(3) \( \mathcal{T} \subset \mathcal{T}' \), \( \mathcal{T} \subset \mathcal{T}'(g) \subset \mathcal{T}(g) \subset \mathcal{T}(n) \)

except for \( g = 1 \) (geometrically).

\[ \text{Def} \quad \text{A knot is \( (n-)solvabale \)} \]
Def: $K \in \mathcal{F}_n \iff$ is ($n$)-solvable

Hence slice knots are ($n$)-solvable.

(1) $H^* (M') \cong H^* (M') \quad \forall k$.

(... some steps...

Note: If $K$ is slice then

(... some steps...)
Other work done at 41,15 year level by

\[ \text{rank } \varphi_{n/0.5} \geq 1. \]

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For each \( n \geq 0, \)
S. Friedl, T. Kim and T. Glimmer.
Other work done at g, 3, 1, 5, level by

\[ \text{rank } a \frac{q_{n+1}}{q_n} \geq 1. \]

For each \( n \geq 0 \),

\( T_{\text{thm}}(\text{cochrane-feichner} \sim 0,n) \),

For \( n = 0, 1, 2, 3, 4, 5, \) contains a \( \mathbb{Z}^2 \).

\( T_{\text{thm}}(\text{livingston}) \) \( \frac{q_1}{q_0}, \frac{q_2}{q_1}, \frac{q_3}{q_2}, \frac{q_4}{q_3}, \frac{q_5}{q_4}, \) contains a \( \mathbb{Z}^2 \).
Thm (Cochran-H-Leidy, 06): For each $n \geq 0,$
$\mathfrak{F}_n/\alpha_{\mathfrak{F}_n}$ has infinite rank.

* In fact we can show our examples are linearly independent of Cochran-Teichner examples that give a $\mathbb{Z}$ in $\mathfrak{F}_n/\alpha_{\mathfrak{F}_n}.$

For this talk, I want to talk about special $\mathbb{Z}^\infty$ subgroups of $\mathfrak{F}_n/\alpha_{\mathfrak{F}_n}$ associated to sequences of prime polynomials.
The string that intersects $D$ into the slice knot

$K \rightarrow D(R,\alpha)(K)$.

Let $R$ be a slice knot, $\alpha$ a curve in $\Sigma$. Let $R(\Sigma)$ be $\alpha = \partial D$, where $D = disk \ in \ Sigma$.

iterated doubling; how to create an $\alpha$-solvable knots.
Example of $D(R,\alpha,\beta,\delta,\xi,\omega)$ (tree foul)

\[ R = \alpha \delta \omega \]

We can iterate the doubling operators.
has a "fractal" structure.

If so (for any \( R^1 \)), then the knot concordance

is \( D(\mathbb{R}, \mathbb{R}) \) injective?

\[ D(\mathbb{R}^1, \mathbb{R}^1) \quad \mathbb{D}(\mathbb{R}^2, \mathbb{R}^2) \quad \mathbb{D}(\mathbb{R}^3, \mathbb{R}^3) \]

Thus we have functions (not homomorphisms)

slice.

Fact: If \( K \) is slice then \( D(\mathbb{R}^1, \mathbb{R}^1)(K) \) is
7-inverse (L-signature defects).

To do this we use Cheeger-Gromov L-

However, in general, it is difficult to
tell if such a knot is \((n+1)\)-solvable.

\[
\sum \mathcal{D}_{(\mathcal{R}, \mathcal{D})} \left( D_{(\mathcal{R}, \mathcal{D})} (K) \right) \in \mathbb{Z},
\]

\[
K = 0 \text{ solvable (K \in \mathbb{Z}).}
\]

Let \( K \) be any knot with \( \text{Art}(K) = 0 \)

Proposition: If \( K \in \mathbb{Z}^+ \) then \( \mathcal{D}_{(\mathcal{R}, \mathcal{D})} (K) \).
In $\mathbb{F}_{p^n}/\mathbb{F}_p$ that survives in subgroup of $\mathbb{F}_{p^n}/\mathbb{F}_p$.

We show that for each $P$, there is a subgroup $\mathfrak{F}_P \subset \mathfrak{F}$, s.t. a subgroup $\mathfrak{F}_P \subset F$. For each sequence $P = (P_l(t), \ldots, P_r(t))$, we associate $\mathfrak{F}_P$. (n)-Solvable filtration
\[ S_1 = \{ \frac{m}{n} \mid m \in \mathbb{Z}, n = \text{odd} \} \]

\[ B = \mathbb{R}/\mathbb{Z} = \overline{\mathbb{R}/\mathbb{Z}} \]

\[ B = \mathbb{R}/\mathbb{Z} = \overline{\mathbb{R}/\mathbb{Z}} \leftarrow B = \mathbb{R}/\mathbb{Z} \]

\[ A = \mathbb{R}/3\mathbb{Z} \leftarrow A = \mathbb{R}/3\mathbb{Z} \]

\[ A = \mathbb{R}/3\mathbb{Z} \leftarrow A = \mathbb{R}/3\mathbb{Z} \]

\[ \exists x : p = 3 \cdot \boxed{a} = 3 \]

\[ A_5 = \mathbb{A}_5 \mathbb{Z}_5 = \mathbb{A}_5 \mathbb{Z}_5 \]

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Let \( S = \{ n \in \mathbb{N} \mid n \neq 0 \text{ and } \text{(p,n) = 1} \} \). Then \( S \) is a localizing system at the prime \( p \).

We can localize \( A \) at the prime \( p \).

Consider any module over \( \mathbb{Z} \) (i.e., abelian group).

If \( A \) is any module over \( \mathbb{Z} \) and \( a \) prime ideal of \( \mathbb{Z} \).

Example of Chen's localization.
Let $\text{a}$ be a right divisor set.

\[ S(\text{a}) = \{g \mid g(a) = \text{a}, \text{a} \in \text{A}, \text{a} \neq \text{e} \} \]

**Prop.**

$A$ is a monomorphism if $\pi : A \to \mathbb{Z}$ is a field of fractions.

(\text{i.e., } \mathbb{Z} \subset \text{A}\text{f of fractions})

poly-torsion-free abelian group

$A$ is an abelian and $P_0$ a $\mathfrak{q}$-field, $\mathfrak{q}$ unit $\mathfrak{g} \neq 0$

$\mathfrak{q} = \{\mathfrak{q} \mid \mathfrak{q} \subset \mathcal{O}[\mathfrak{t}, t], \mathfrak{g} \neq \text{unit} \}$

Consider:

Non-commutative localization at a prime
otherwise, we say they are strongly coprime, \( (p, q) = 1 \).

\[ \text{mean} \rightarrow \text{cp} \rightarrow R^p = R^q \]

and some \( R^p \) of \( R^p \) and \( R^q \) of \( R^q \), and some denoted \( (p, q) \neq 1 \) for some non-zero roots,

\[ \text{Def: We say \( (p, q) \) and \( R^p \) and \( R^q \) are isogenous,} \]

that are "coprime" to a fixed \( \text{p} \).

We would like to invert all "polynomials" \( \mathbb{F} \rightarrow \mathbb{F} \).  

Hence we can invert the set \( S = S(0) \)
\[
\begin{align*}
(p_k, p_0) &= 1 \quad \text{when } k \neq 0, \\
\{ \frac{K}{K+1}, \frac{K+1}{K+2} \} \quad \text{roots } \quad R_{k, n} = \{ K \} \\
\mathbb{R}_k^+ \quad \text{for } \quad K \in \mathbb{R}^+ \\
K_{(t)} = (K_t - (K+1))(K+1) + K_t - K, \quad K \in \mathbb{R}^+ \\
\text{roots of } g \quad \text{root of } p \\
\text{"h" root of } g \\
S = 2 \quad \text{Sina } 2 \neq I \quad \text{Sina have no common root}.
\end{align*}
\]
\[ M_{\gamma} = W_{\Omega_0\cdot s} \]  

\[ M \text{ is a (right) } \Omega_0\text{-module, then} \]

\[ s^p = s \left( g \in \Omega_0 \mid (p, q) = 1 \right) \]

For \( p(t) \) (non-unital, non-zero) define \( p(b) \) in \( \Omega_0 \).

If \( p(a) \) is relatively prime to \( \text{group } F, \) and nonzero \( \mathbb{A}, b \in F \) for any \( f \), \( g \), \( \text{free abelian} \)

\[ (p, q) = 1 \]

\[ \text{Prop: } p, q \in \Omega_0 \text{ (non-zero, non-unit,} t \)
\[ p = \{ p_1, \ldots, p_n \} \]

We can use this localization to define

\[ f(p) = \begin{cases} 1 & \text{if } \bigcap_{i=1}^{g(a)} S^{p_i} = 0 \end{cases} \]

\[ \overline{\bigcup \{ \text{for any } a \notin \Omega \}} \]
Let $P = \{p_1, \ldots, p_n\}$. For each $1 \leq l \leq n$,

\[
\left[\text{derived}\right]
\]
A knot $K$ is $(n', p')$-solvable if
\[ \mathcal{H}^g_1(M') \cap \mathcal{H}^g_1((\mathbb{Z}/p')M') = \mathcal{H}^g_1(M') \]
except for $g = 1$ (geometrically). All surfaces (with trivial normal bundle) all disjoint surfaces (with trivial normal bundle) all disjointly embedded have a basis $\{ g_i \}_{i=1}^g$ of embedded $\mathbb{Z}^2$.

\[ H^2(W) \cong H^1(M') \cong H^1((n', p')M') \]

Let $W$ (n-solution) s.t.

$W$ (0-surgery on $K$) bounds a spin $M^k$ if A knot $K$ is $(n', p')$-solvable (neither $n'$ nor $p'$ is divisible by 2).
Hence $Z^d \cap Z^g = 0$.

And $\operatorname{Hom}(Z^d, Z^p)$ is a monomorphism.

If $Z^d \cong Z^p / \mathfrak{p}$, then the image of $Z^d$ under $\mathfrak{f} \to \mathfrak{f}^{1+u}$ is 0.

If $\mathfrak{f} = (p_1, \ldots, p_n)$, then the image of $\mathfrak{f}^{1+u}$ is strongly Cohen-Macaulay.

There is a subgroup $\mathbb{Z}^d \subseteq \mathbb{Z}^{\infty}$.

Theorem (Cohen-Keisler): For each $F = \{p_1, \ldots, p_n\}$,
Let $R^t = p^t(k)$ then

$$\forall x, y \in R^t : (x, y) = (p_1, \ldots, p_m) \Rightarrow m \leq k.$$
and \( \mathbb{Z} \neq \mathbb{Z} \cap \mathbb{P} \). When \( (\ldots, m_0, \ldots) \neq (\ldots, m_0, \ldots) \)

Then \( \mathbb{Z} \cap \mathbb{P} = \mathbb{N} \) if \( \mathbb{Z} \) is an \( \mathbb{N} \)-module.

Define \( \ker(K, \ldots, m_n) = \mathbb{D}^{(\text{rank}(K, \ldots, m_n))} \) \( \ker(K, \ldots, m_n) \) over \( \mathbb{Z} \).

Let \( \{K_i\} \) be a infinite set of knots with