An overview of knot and link concordance from the viewpoint of the n-solvable filtration

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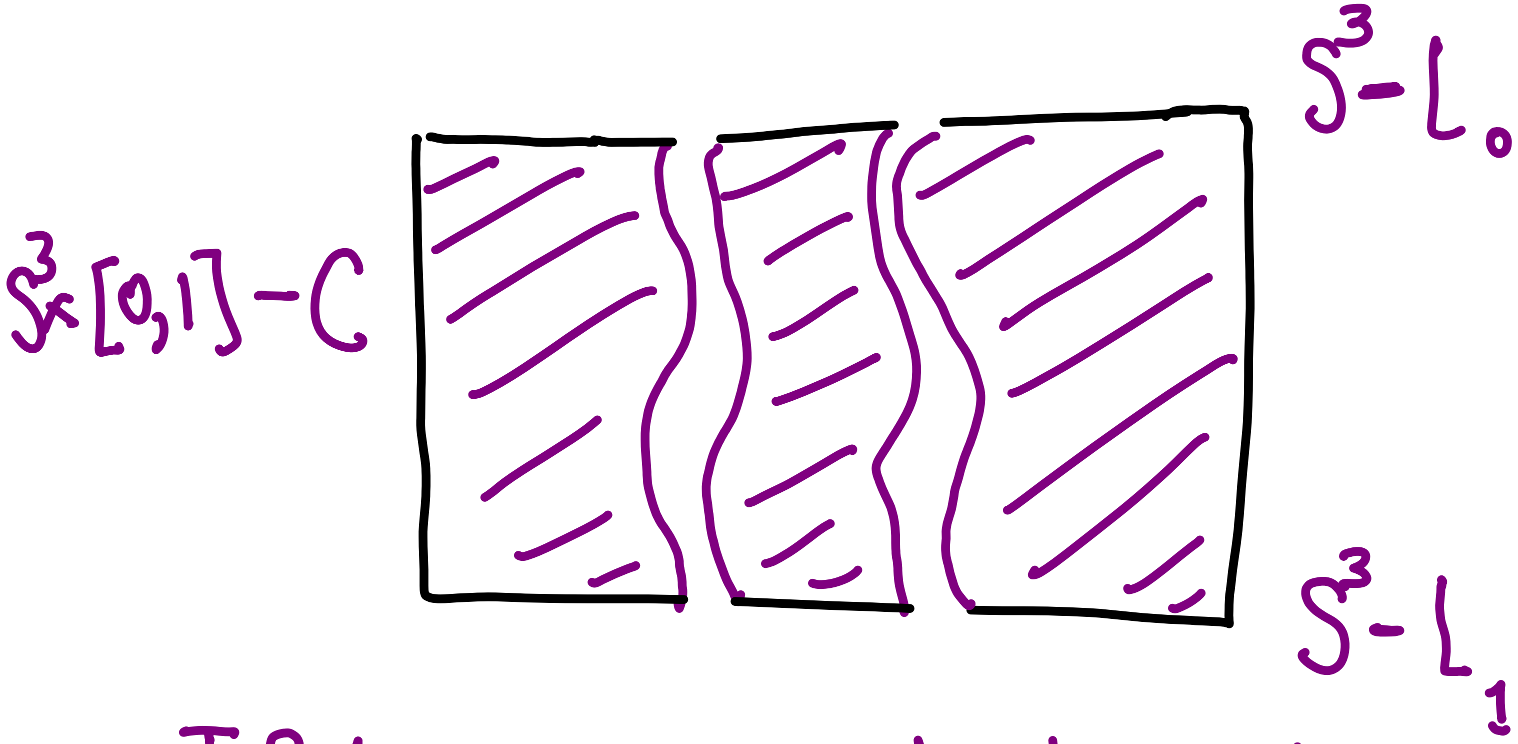
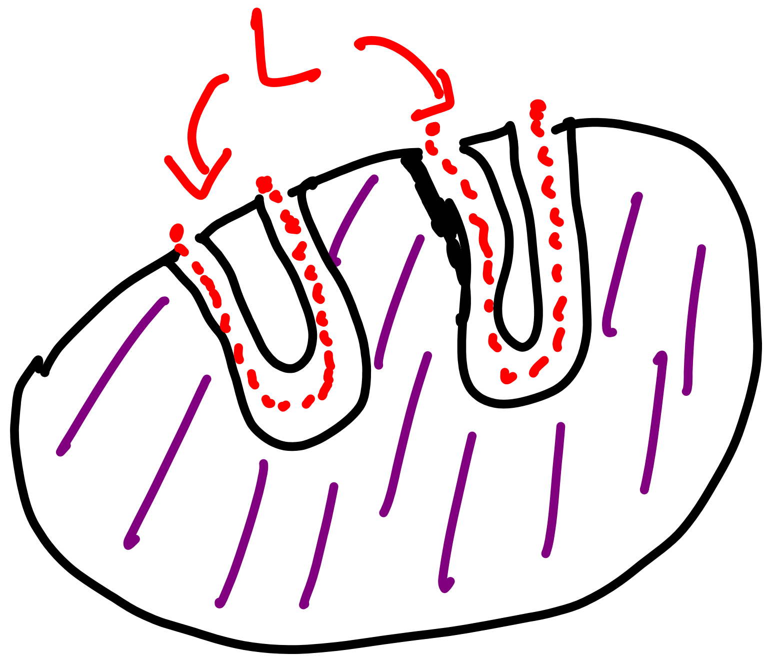
Knot and Link concordance (almost) purely a question about homology cobordism of associated 3-manifolds (zero surgery, branched covers, ±1 surgeries).



If L_0 is concordant to L_1 then $S^3\setminus L_0$ is homology cobordant rel ∂ to $S^3\setminus L_1$, and so same for any surgeries on L_i.

<u>Lemma</u>: If link L (0 linking numbers) is a slice link then zero-framed Dehn surgery M_L is the oriented boundary of a compact 4-Md W s.t.

1. $H_1(M_L) \xrightarrow{\cong} H_1(W)$
2. $H_2(W) \cong 0$
3. $\pi_1(W)$ normally gen't by $\pi_1(M_L)$

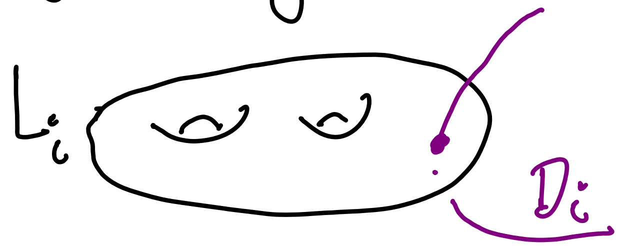


$W = B^4 -$ slice disks

Definition: (C-Orr-Teichner 1998) L is n-solvable if $M_L = \partial W^4 \rightarrow$ SMOOTH

1. $H_1(M_L) \xrightarrow{\cong} H_1(W) \cong \mathbb{Z}^{\#\text{-comps}}$

2. $H_2(W;\mathbb{Z})$ has a basis of embedded surfaces $\{L_i, D_i\}$ with product neighbhds all disjoint except $L_i \cdot D_i = 1$ geometrically



so intersection form is $\oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3. $\pi_1(L_i) \subseteq \pi_1(W)^{(n)}, \pi_1(D_i) \leq \pi_1(W)^{(n)}$

n.5-solvable if $\pi_1(L_i) \leq \pi_1(W)^{(n+1)}$

$\mathcal{C}$ = Smooth Knot (or String Link) Concordance group

$\mathcal{F}_n$ = subgroup of n-solvable knots

<u>Prop (COT)</u>: This induces a filtration by subgps

$$0 \leq \ldots \mathcal{F}_{n.5} \leq \mathcal{F}_n \ldots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_{.5} \leq \mathcal{F}_0 \leq \mathcal{C}$$

EQUIVALENT DEF OF n-solvable

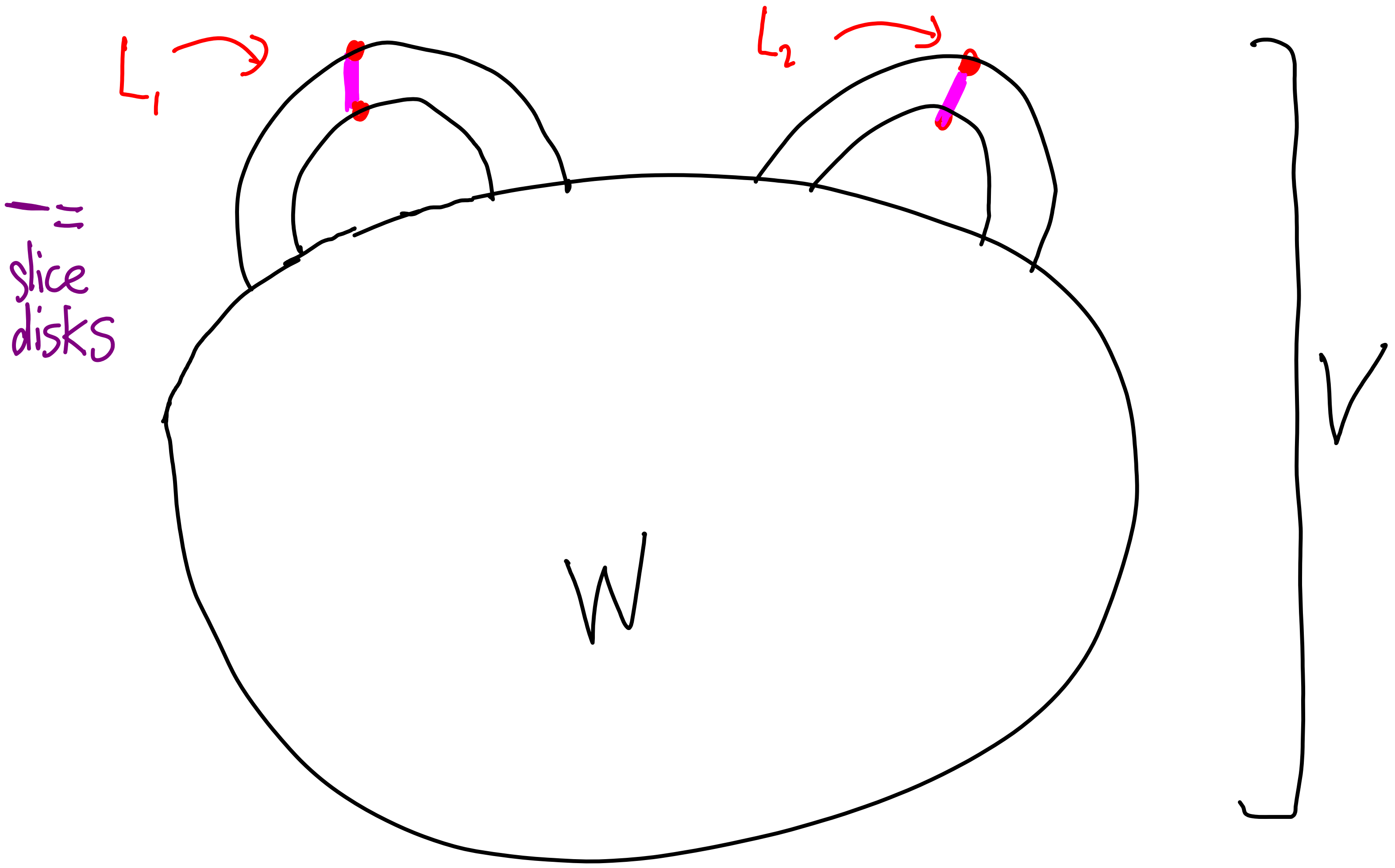
$L \hookrightarrow S^3$ is slice in a smooth 4-mfd V s.t.

1. $H_*(V) \cong H_*(\#S^2\times S^2 - B^4)$
2. $H_2(V)$ has a basis in $V - \amalg$ slice disks

$\{L_i, D_i\}$ as in def. of n-solvable

$\pi_1(L_i)$ and $\pi_1(D_i) \subseteq \pi_1(V - \amalg \text{ slice disks})^{(n)}$

Proof: $V = W \cup$ 2-handles on meridians



$\partial V = S^3$ $\partial W = M_L$

What is Known about $\{\mathcal{F}_n\}$

FOR KNOTS:

1. $\mathcal{F}_0$ = Arf invt. zero Knots, $\mathcal{F}_{.5}$ = Algebraically slice Knots, Knots in $F_{1.5}$ have 0 Casson-Gordon invts.

2. integral n, $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \subseteq \mathcal{F}_n / \mathcal{F}_{n.5}$
(Milnor, Tristram, COT, Livingston, C-Harvey-Leidy)

3. evidence of "p-primary decomposition"
(Levine, S-G Kim, Taehee Kim, C-H-L)

4. {Topologically Slice Knots} $\subseteq \bigcap_{\text{all } n} \mathcal{F}_n$

For Links

1. modulo local knotting $\mathbb{Z}^{\infty} \subseteq \mathcal{F}_n / \mathcal{F}_{n.5}$

(Harvey, more structure: Cha, Harvey - Cochran)

OPEN QUESTIONS (more later)

OTTO

1. "other half" of filtration $F_{n.5}/F_n \overset{?}{=} 0 \quad n \geq 0$

2. 4 or 8 torsion in $F_n/F_{n.5}$??

3. Does $\bigcap_{n=0}^{\infty} F_n = \{\text{Topologically slice links}\}$?

HARVEY

4. $F_n/F_{n.5} \cong \bigoplus_{p(t)}$?

OTTO

5. for links what is relationship with Milnor's $\bar{\mu}$ invts ?

6. Does $\mathcal{F}_n = \mathcal{G}_{n+2}$? grope filtration of $\mathcal{C}$

7. extension problems $0 \to \frac{\mathcal{F}_{.5}}{\mathcal{F}_{1.5}} \to \mathcal{C}/\mathcal{F}_{1.5} \to \frac{\mathcal{C}}{\mathcal{F}_{.5}} \to 0$

OTTO 8. For links $\mathcal{C}$ is not abelian, Is $\mathcal{F}_n/\mathcal{F}_{n.5}$ abelian?

Cochran 9. Are there similar meaningful filtrations for
Ruberman Topologically slice Knots?

Powell 10. "Classify" $\mathcal{F}_{1.0}/\mathcal{F}_{1.5}$ higher algebraic

concordance groups

How to create n-solvable Knots : Satellites, Infection, Generalized Doubling Operators

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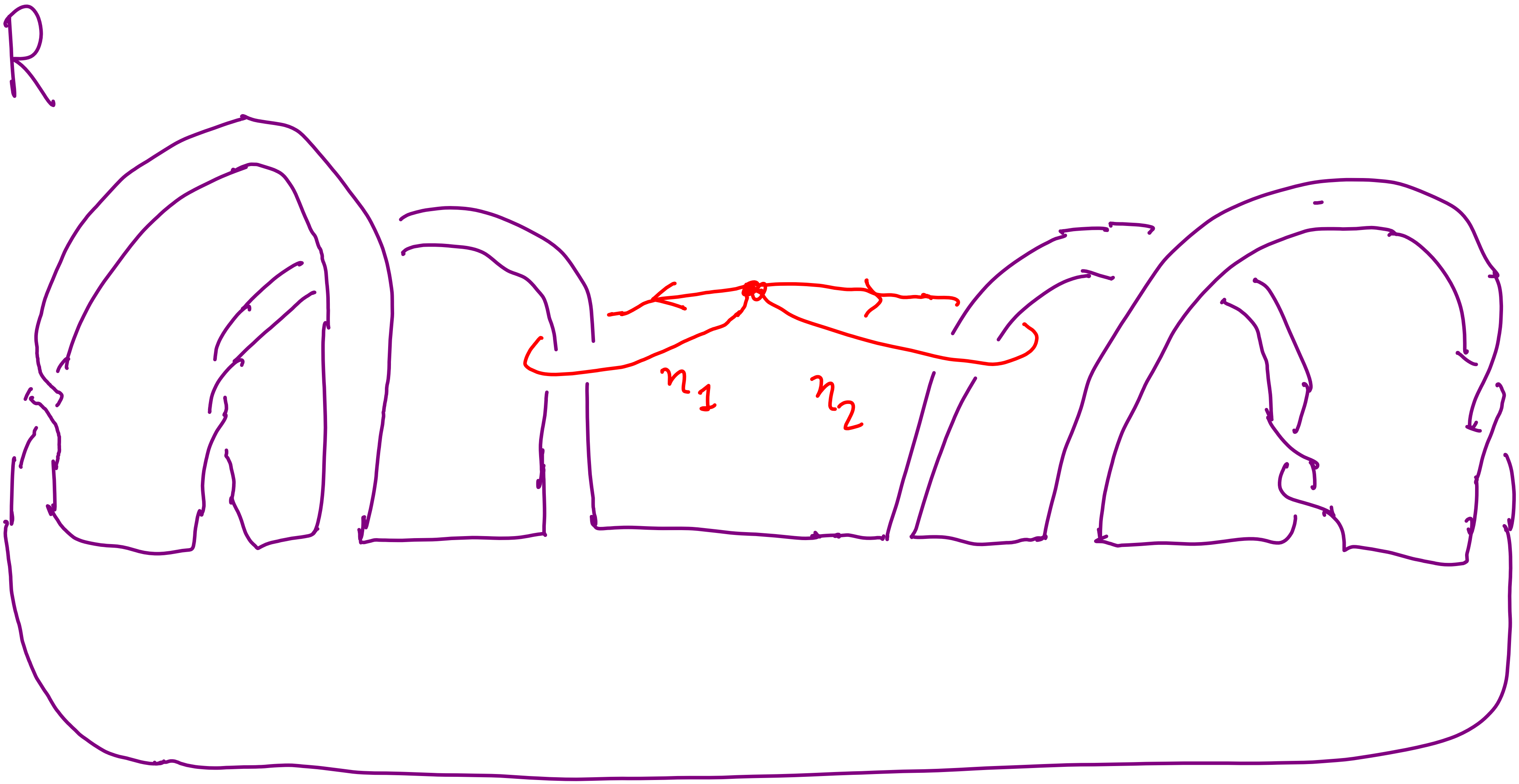
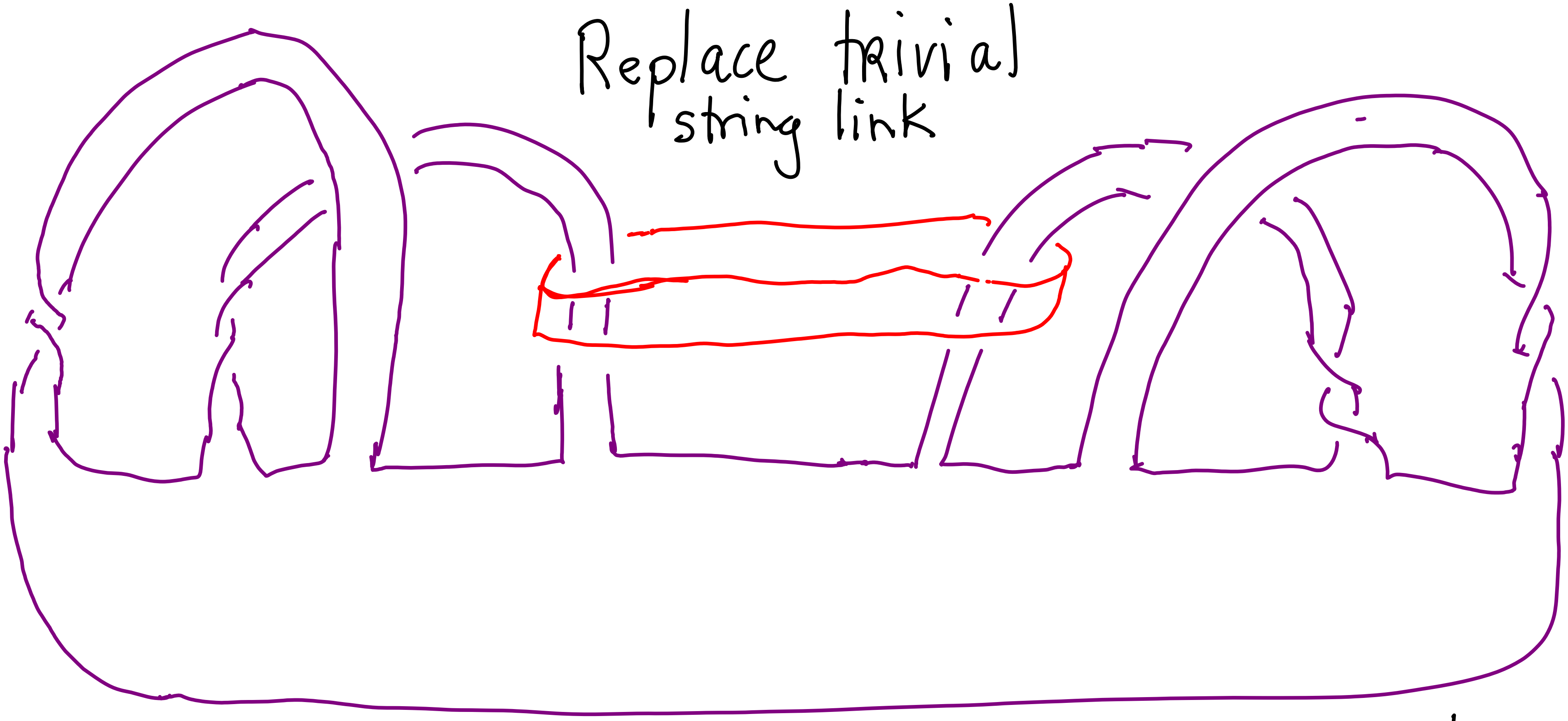
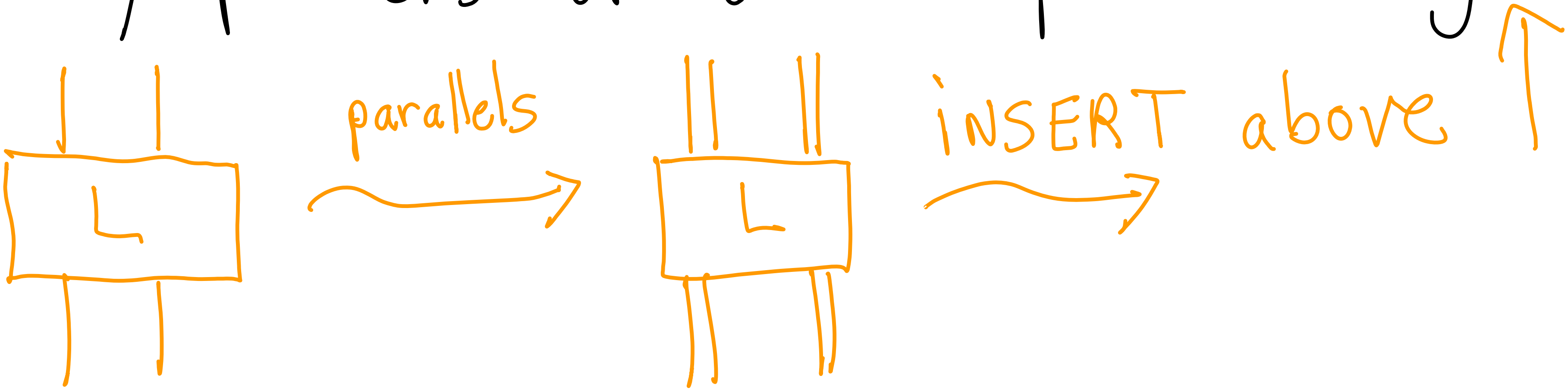
Start with a ribbon Knot R and "infecting curves $\{\eta_1, \eta_2\}$ with $lk(\eta_i, R) = 0$.



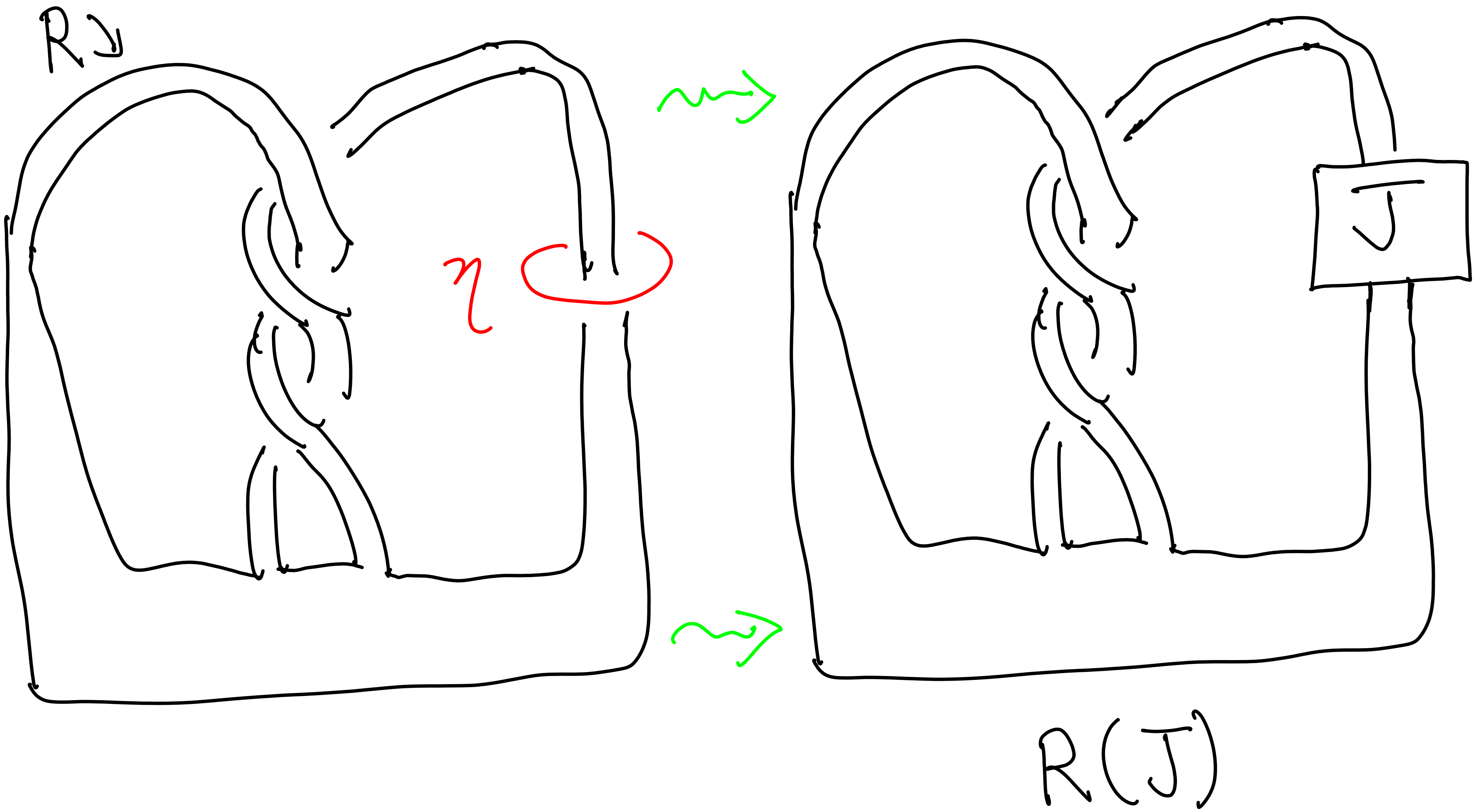
Replace trivial string link



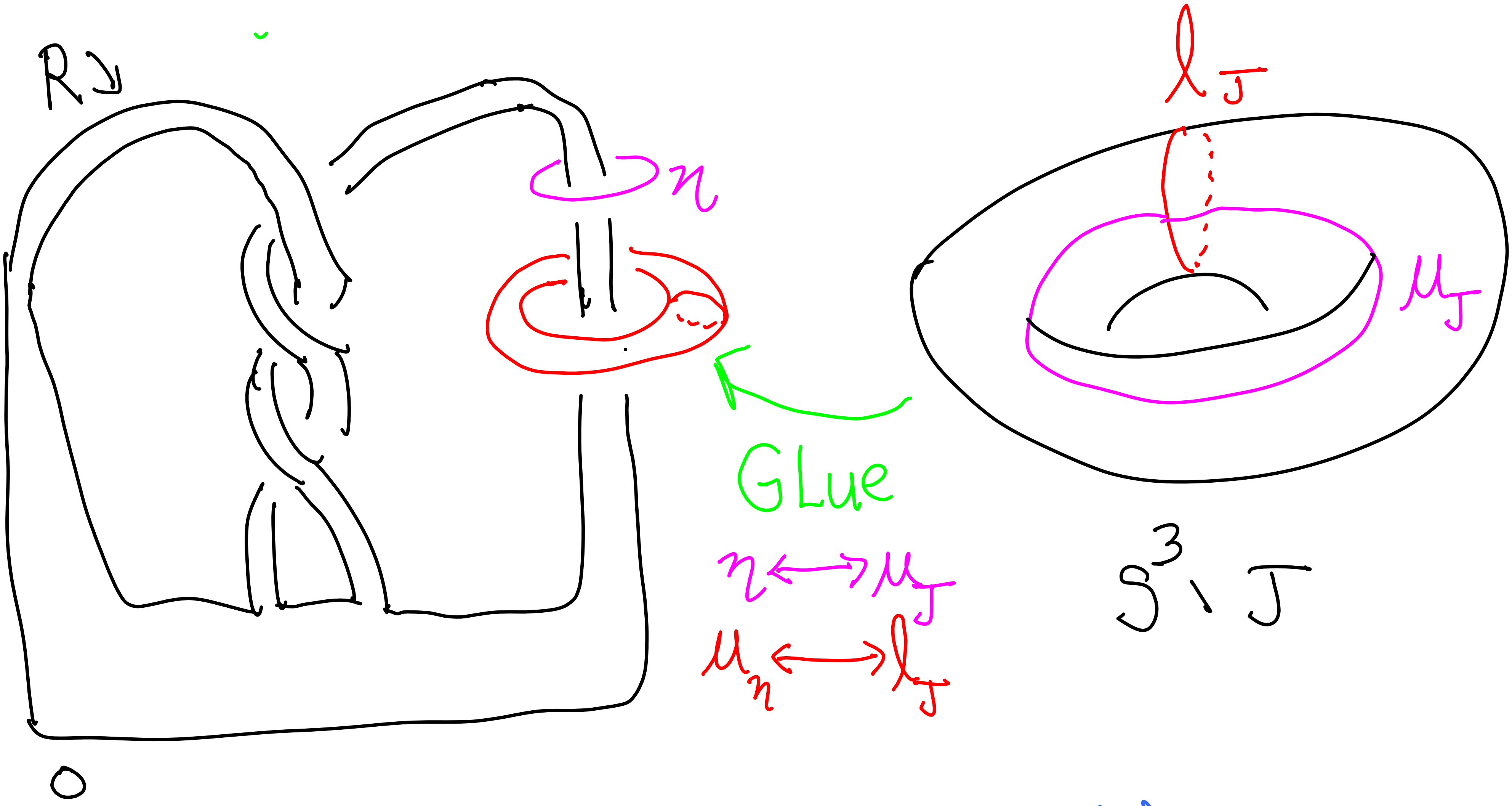
by parallels of a 2-component string link



Restrict to case where L is a knot J and call result $R(J,\eta)$: precisely a satellite of J with winding number zero:



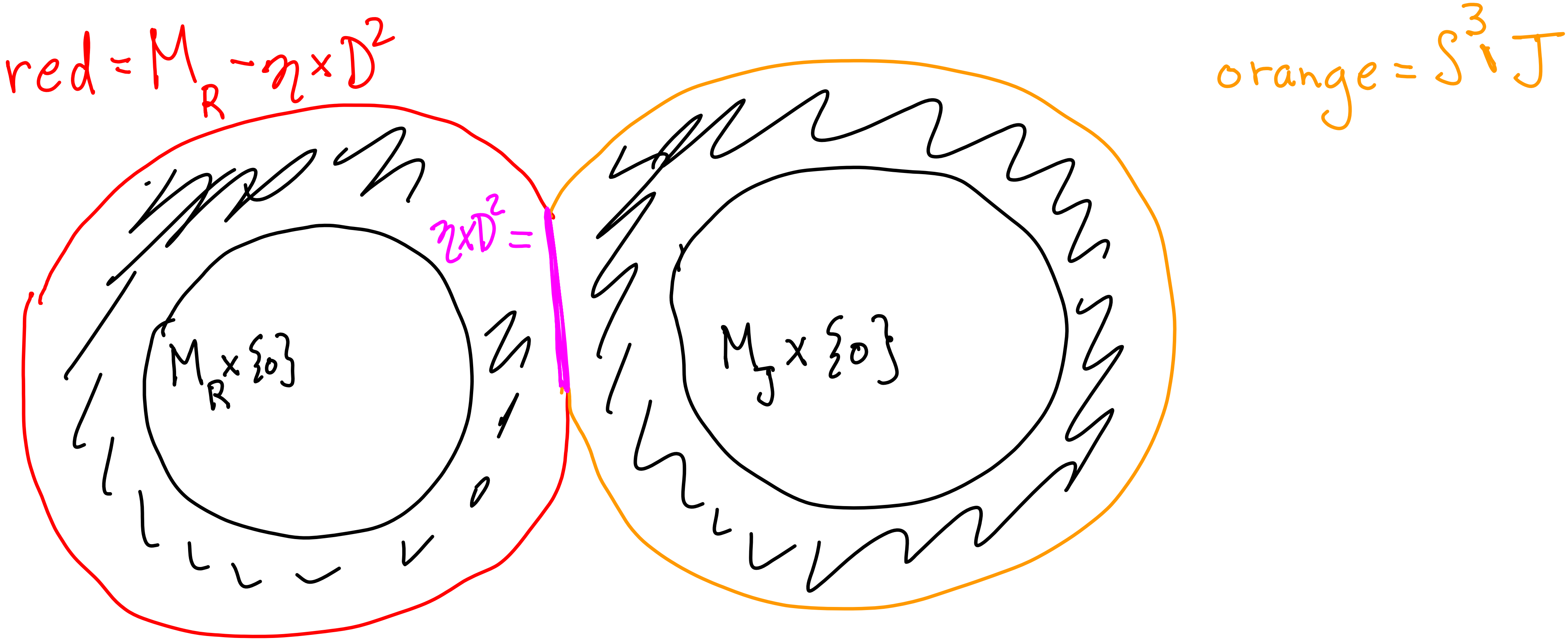
Exercise: same as replace $\eta \times D^2$ by $S^3 \setminus J$



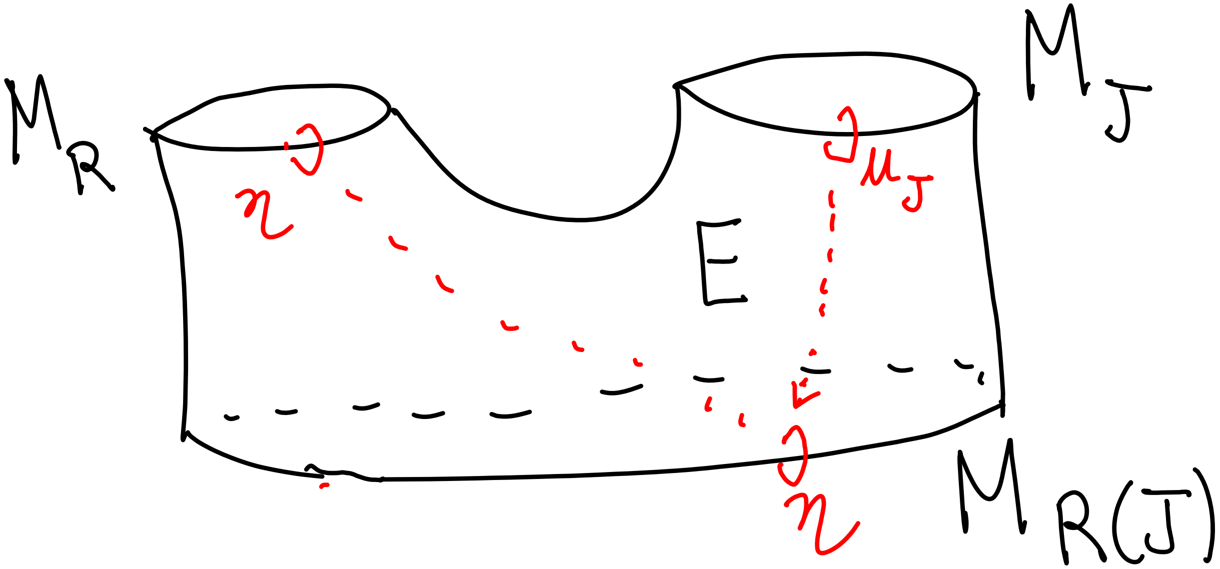
$* \pi_1(S^3 \setminus J) \subseteq \pi_1(M_{R(J)})^{(1)}$

Prop: (CHL, COT) If J is n-solvable then $R(J,\eta)$ is $(n+1)$-solvable.

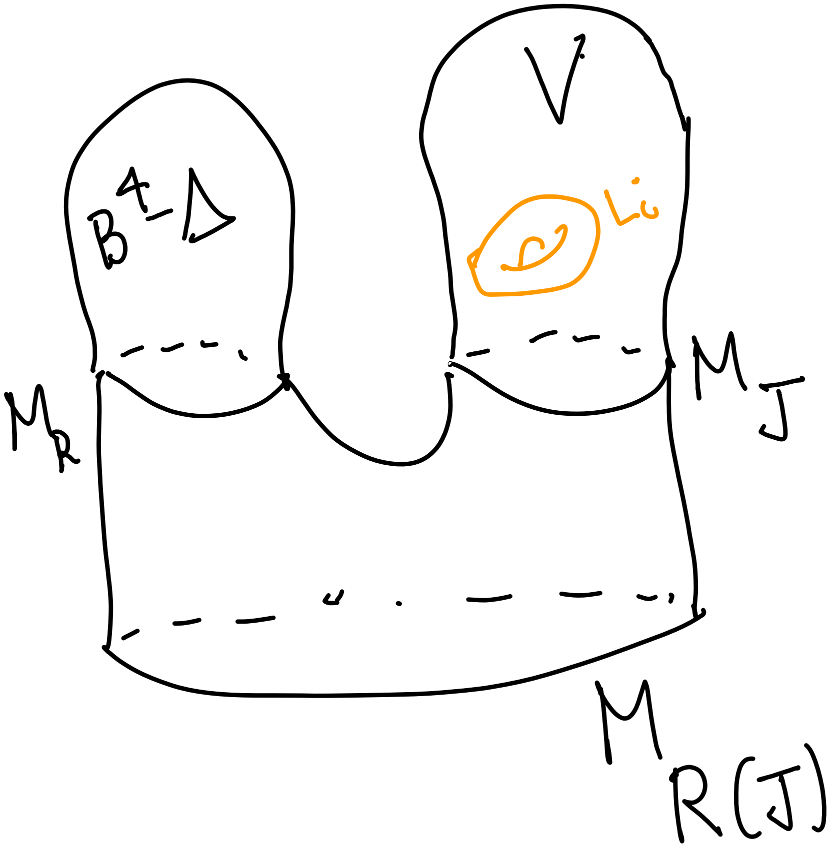
Proof uses **Fundamental Cobordism** E



$$M_R \times [0,1] \cup M_J \times [0,1] \equiv E$$



$\pi_1(M_J) \subseteq \pi_1(E)^{(1)}$



Glue on $B^4 - \Delta \rightarrow$ slice disk for R

V n-solution for J

Result is $(n+1)$-solution for J

$\therefore$ For any such (R_i, η_i) $i = 1, \ldots, n$ and any Arf invt. 0 Knot J, $R_n \circ \cdots \circ R_2 \circ R_1(J) \in F_n$

Idea of how we show $F_n / F_{n.5}$ is ∞-gen. and exhibits p-primary decomposition

Idea: vary R_i, J to get lin. independent Knots (what happens if vary η_i? Franklin)

distinguish by "signature defects".

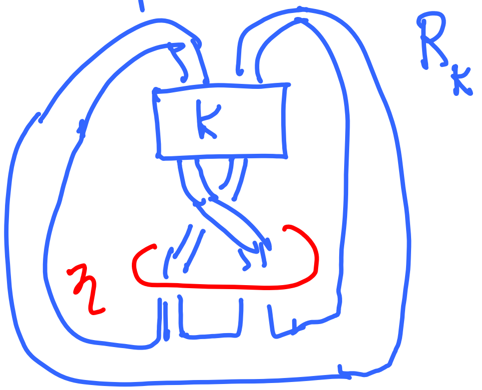
Aside: Can view (R,η) as an operator

$$\begin{tikzcd} \mathcal{C} \arrow[r] & \mathcal{C} \\ J \arrow[r] & R(J,\eta) \end{tikzcd}$$

Conjecture: For "robust (R,η)" this operator is injective and so $\mathcal{C}$ is a fractal set

Moreover "different" R and η have distinct images.

Example: • Whitehead double operator



• These operators

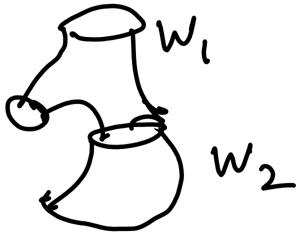
Signature Defects

(Hirzebruch, Atiyah, APS)

Def: A signature defect is an invariant of regular coverings of 3-manifolds that arises from an invariant of reg. covers of 4-mfds.

Suppose $\sigma_\Gamma : \left\{ \begin{matrix} \tilde{W}^4 \\ \downarrow \\ W^4 \end{matrix} \ \Gamma\text{-cover} \right\} \longrightarrow \mathbb{R}$ is such that:

1. if W closed $\sigma_\Gamma(\tilde{W}) = \sigma(W)$ or just some function of W!
2. $\sigma_\Gamma(-W) = -\sigma_\Gamma(W)$
3. Novikov Additivity



Then define $\rho_\Gamma\left(\begin{array}{c}\tilde{M}^3\\ \downarrow\\ M\end{array}\right) = \sigma_\Gamma\left(\begin{array}{c}\tilde{W}\\ \downarrow\\ W\end{array}\right) - \sigma(W)$

for some $\partial\left(\begin{array}{c}\tilde{W}\\ \downarrow\\ W^4\end{array}\right) = \begin{array}{c}\tilde{M}\\ \downarrow\\ M^3\end{array}$

easy: conditions ensure it is independent of W

<u>Examples</u>: 1. Γ finite group $\sigma_\Gamma = \dfrac{\sigma(\tilde{W})}{|\Gamma|}$ Hirzebruch

using "signature of closed mfds multiplies in covers"

2. $\Gamma = \mathbb{Z}_{p^r}$ $\quad \sigma_\Gamma(\widetilde{W}) =$ signature of an eigenspace of $\tau_* : H_2(\widetilde{W};\mathbb{C}) \circlearrowleft$

Rochlin, Neumann, Casson-Gordon

3. Choose rep: $\Gamma \xrightarrow{\phi} U(n) \le \mathrm{Aut}\,\mathbb{C}^n$

$$\sigma_\Gamma(\widetilde{W}) = \text{signature of } H_2(W;\mathbb{C}^n_\phi)$$

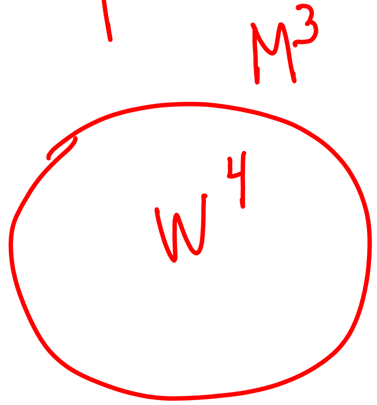
(APS)

4. Use left-regular rep. $\Gamma \xrightarrow{\ell_\Gamma} U(\ell^2\Gamma)$

$$\sigma_\Gamma^{(2)}(\widetilde{W}) = L^{(2)} \text{ or Von-Neumann signature}$$

on $H_2(\widetilde{W}) / \mathbb{Z}\Gamma$-torsion.

In particular given



$$\begin{tikzcd}
\pi_1(M^3) \arrow[r, "\phi"] \arrow[d, "i_*"] & \Gamma \\
\pi_1(W^4) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

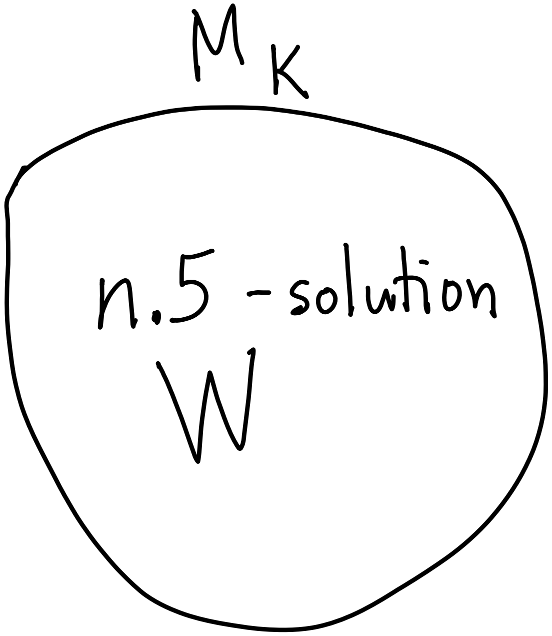
$$\boxed{\begin{array}{l} \rho(M,\phi) \equiv \sigma_{\Gamma}^{(2)}(W,\psi) - \sigma(W) \\ \text{Cheeger-Gromov or von Neumann } \rho\text{-invariant} \end{array}}$$

2 Key features not shared by other sign. defects:

A. If $\pi_1(M) \longrightarrow G \hookrightarrow \Gamma$ then $\rho(M,G) = \rho(M,\Gamma)$

B. $\rho(M_K, \text{abelianization}) = \int \sigma_\omega(K) \in \mathbb{R}$

FIRST Fundamental RESULT of C-Orr-Teichner:



$$\begin{tikzcd}
\pi_1(M_K) \arrow[rr, "\phi"] \arrow[d, "i_*"] & & \Gamma \\
\pi_1(W) \arrow[r] & \pi_1(W)/\pi_1(W)^{(n+1)} \arrow[ur] &
\end{tikzcd}$$

then $\rho(M,\phi) = 0 - 0 = 0$ for any Γ

that is PTFA, poly-(torsion-free abelian)

for example if $\Gamma = \pi_1(W)/\pi_1(W)^{(n+1)}_r$

Sketch of Proof in case M_K is $(n+1)$-solvable:

By hypothesis $\pi_1(L_i)$, $\pi_1(D_i)$ lie in Kernel $\phi : \pi_1(W) \to \Gamma$ so L_i, D_i lift to $\widetilde{W}$

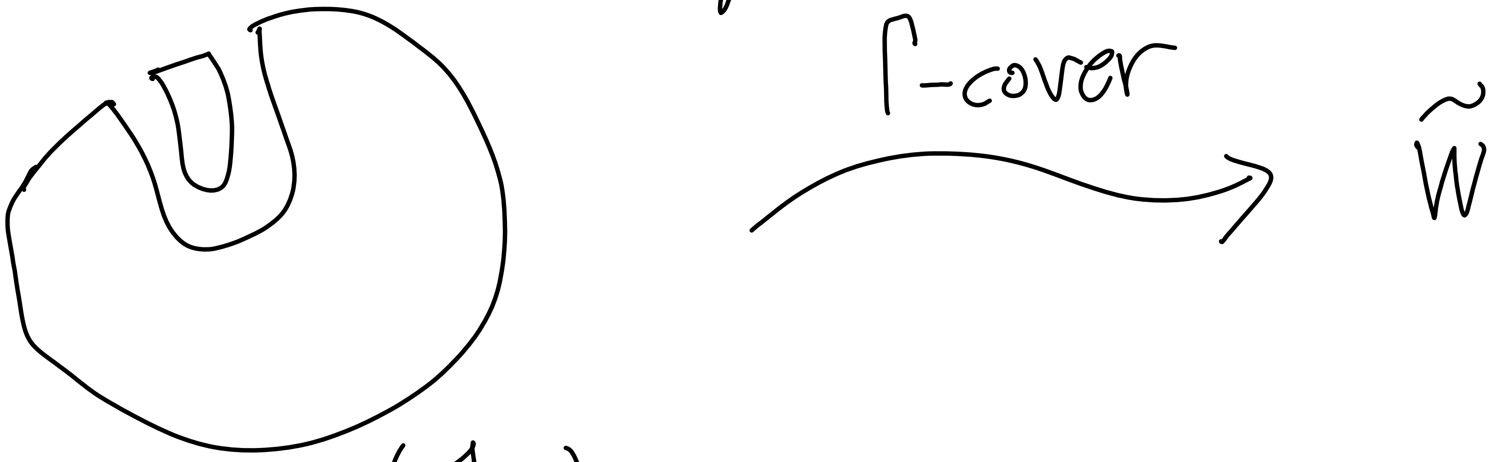
$$\therefore H_2(\widetilde{W}) \underset{\mathbb{Z}\Gamma\text{-module}}{\cong} \bigoplus \underset{\widetilde{L}_i, \widetilde{D}_i}{\underset{\uparrow}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}} \oplus ?$$

Homological Algebra $\Longrightarrow$? is $\mathbb{Z}\Gamma$-torsion

$$\therefore \sigma_\Gamma^{(2)}(\widetilde{W}) = 0.$$

END OF SKETCH

Aside: This homological algebra is crucial.
It concerns the question:



$H_2(W) = H_2(B^4 - \Delta) = 0$

For what groups Γ is $H_2(\widetilde{W}) = 0$ or "Torsion"

Answer: (J. Adams, Rochlin, Howie, Strebel)
Γ PTFA or locally indicable, Γ finite p-group

Sketch of proof that ${F_n}/{F_{n.5}}$ is large.

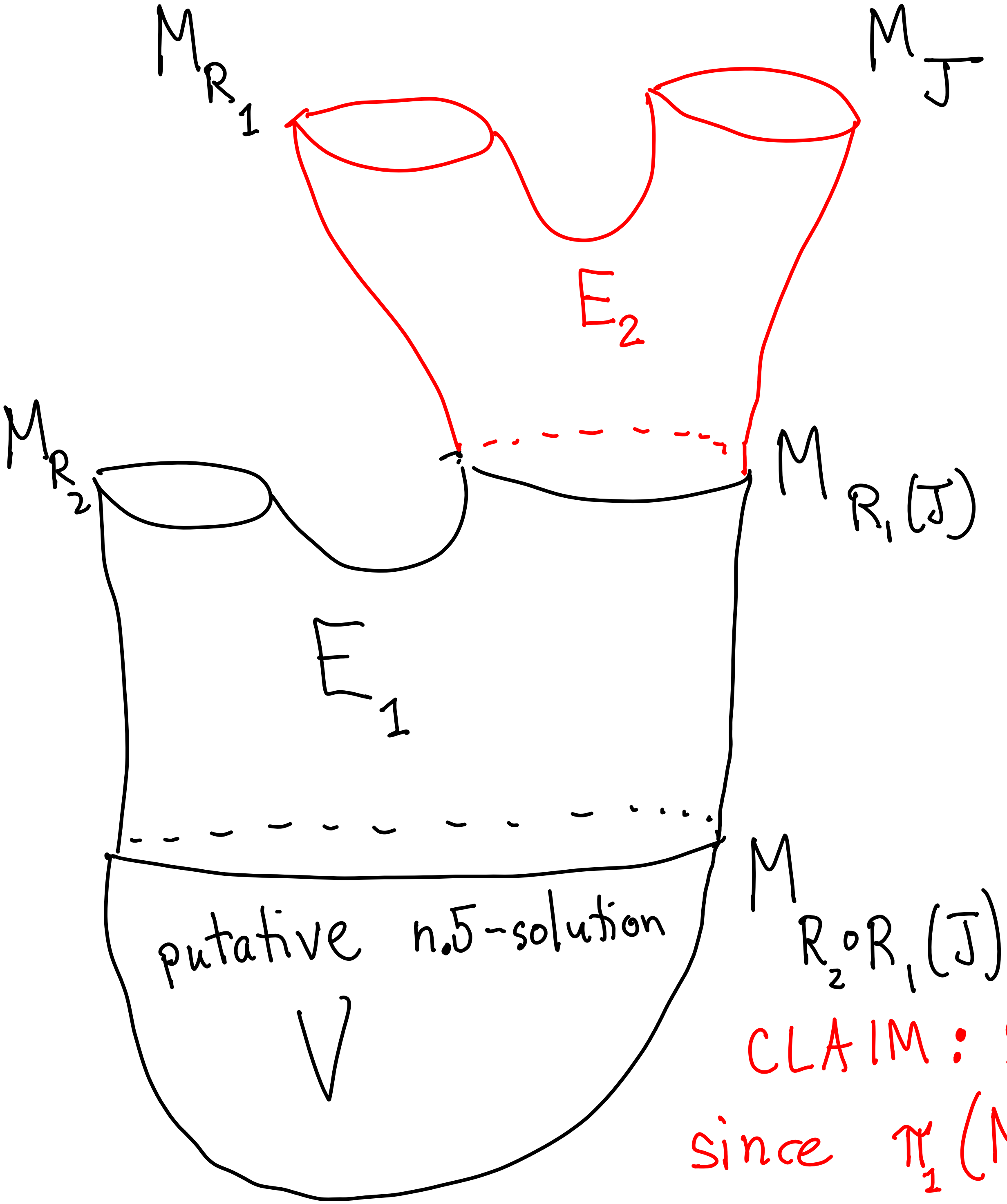
Consider $R_n \circ R_{n-1} \circ \cdots \circ R_1(J)$ for varying R_i, J. Show lin. ind. under some conditions.

First consider one such iterated infection and show its not n.5-solvable

Suppose it IS n.5-solvable:

CONSIDER A COBORDISM W

$\left(\begin{matrix} n=2 \\ \text{shown} \end{matrix}\right)$



$$\begin{tikzcd}
\pi_1(M_J) \arrow[d, "i_*"'] \arrow[dr, dashed, "\phi_J"] & \\
\pi_1(W) \arrow[r, "\phi"] & \dfrac{\pi_1(W)}{\pi_1(W)^{(n+1)}_r} \arrow[d, phantom, "\parallel"] \\
& \Gamma
\end{tikzcd}$$

CLAIM: ϕ_J factors thru $\mathbb{Z}$

since $\pi_1(M_J) \subseteq \pi_1(E_2)^{(1)} \subseteq \pi_1(W)^{(n)}$

By definition: $\rho(\partial W, \phi) = \sigma_{\Gamma}^{(2)}(W) - \sigma(W)$

By Fund. Thm. $\rho(\partial W, \phi) = 0$

so $\rho(M_J, \phi_J) = \underbrace{- \sum_{i=1}^{n} \rho(M_{R_i}, \phi_i)}_{\text{universally bounded given } R_i}$

$\therefore$ if $|\rho(M_J, \phi_J)|$ very large $\Rightarrow$ contradiction

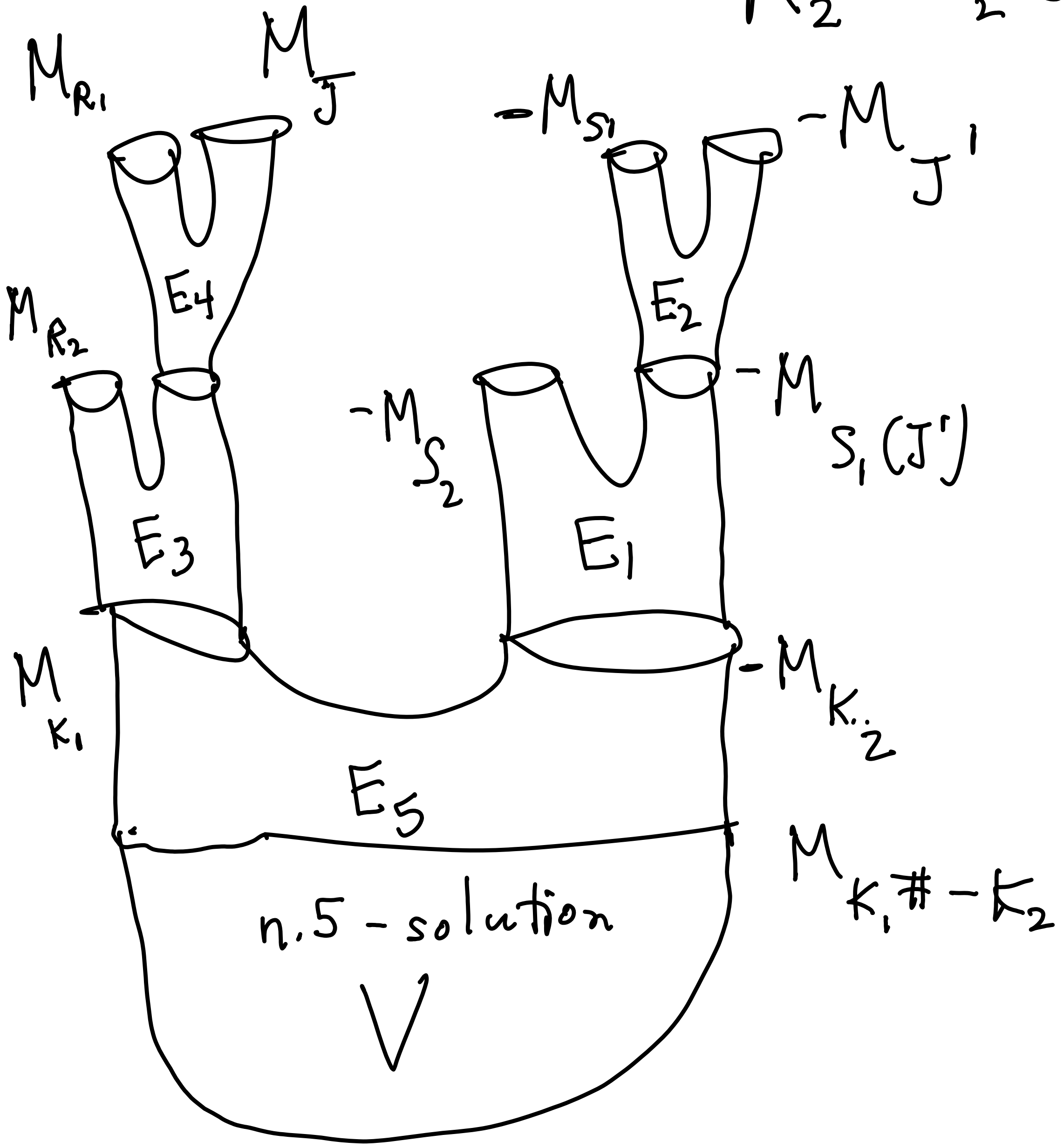
but $\phi_J : \pi_1(M_J) \twoheadrightarrow \mathbb{Z} \xrightarrow{i} \Gamma$

If i non-zero $\rho(M_J, \phi_J) = \rho(M_J; \pi_1(M_J) \twoheadrightarrow \mathbb{Z})$

$= \rho_0(J) = \int \text{signatures } \overline{J}$

How do we show linear independence for different R_i? Suppose $K_1 = R_2 \circ R_1(J)$ concordant to

$K_2 = S_2 \circ S_1(J')$

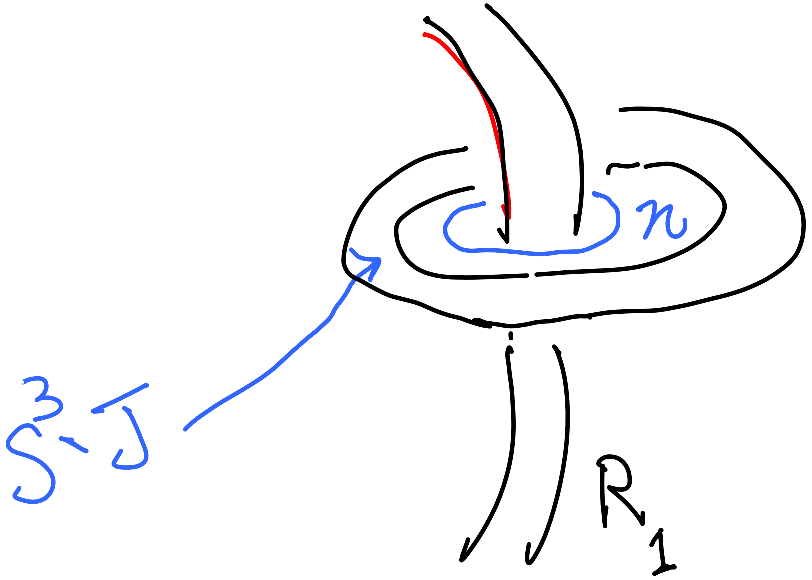


$$\begin{tikzcd}
\pi_1(W) \arrow[r] \arrow[rr, bend left, "\phi"] & \dfrac{\pi_1(W)}{\pi_1(W)^{(n+1)}} \arrow[r, two heads] & \dfrac{\pi_1(W)}{??}
\end{tikzcd}$$

Use different coefficient systems (PTFA)

We want $\phi(\pi_1(M_J)) \neq 0$ $\phi(\pi_1(M_{J'})) = 0$

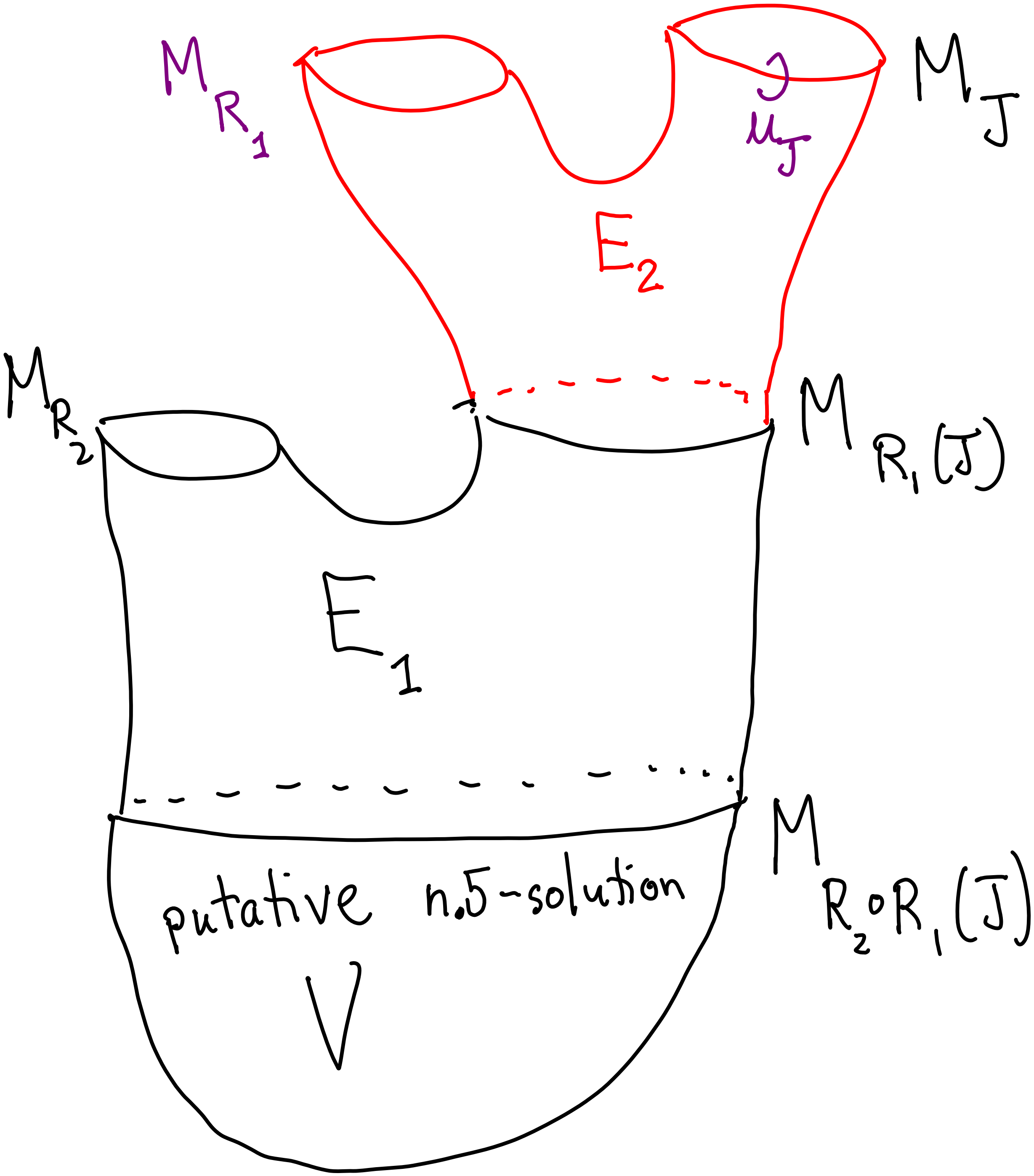
$\phi(\mu_J] \neq 0$ $\phi(\mu_{J'}) = 0$



But $\mu_J \equiv \eta \subseteq M_{R_1}$

and η is $\Delta_1(t)$ torsion in Alexander module of R_1 where Δ_1 = Alexander polynomial of R_1. So localization can kill torsion, hence η.

· Hardest PART : Show $\phi_J(\mu_J) \neq 0$ $\mu_J \notin \pi_1(W)^{(n+1)}$

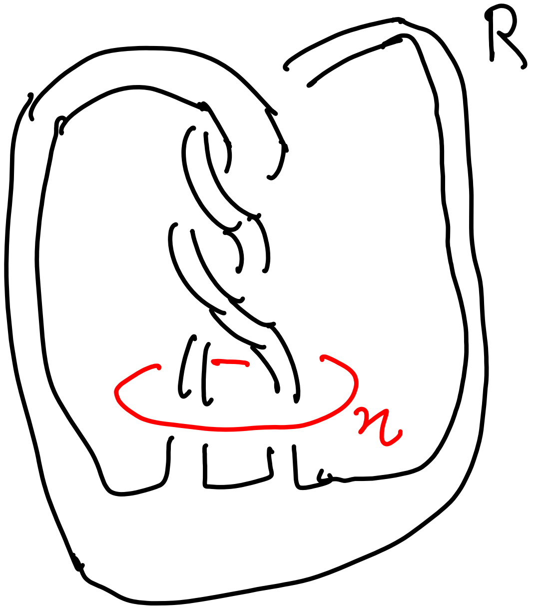


$$\begin{tikzcd}
\pi_1(M_J) \arrow[d, "i_*"'] \arrow[dr, dashed, "\phi_J"] & \\
\pi_1(W) \arrow[r, "\phi"] & \frac{\pi_1(W)}{\pi_1(W)^{(n+1)}_r} \arrow[d, equal] \\
 & \Gamma
\end{tikzcd}$$

Easiest Case:

$n=1$ How do we show $\eta \neq 0$ under any map

$$\pi_1(M_R)^{(1)} \longrightarrow \frac{\pi_1(W)^{(1)}}{\pi_1(W)^{(2)}}$$



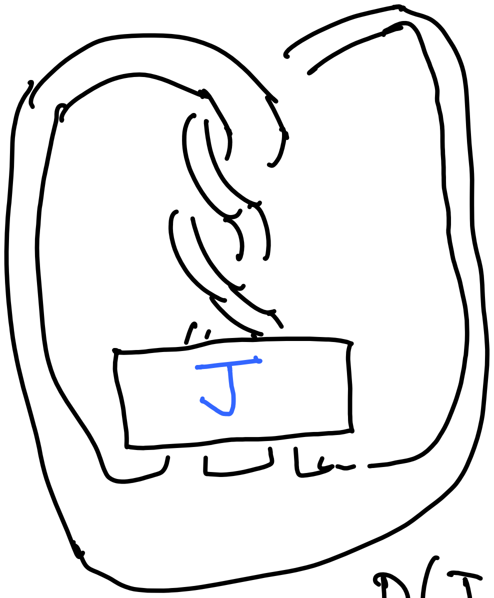
R

This is question of Alexander modules:

$$\eta \in A_0(R) \xrightarrow{j_*} A_0(W)$$

same

unknown slice disk exterior or 1.5-solution



R(J,η)

Use analogs of classical result:

$$\operatorname{Ker}\left(i_*: A_0(K) \longrightarrow A_0\left(\substack{\text{slice disk}\\ \text{exterior}}\right)\right)$$

is <u>Lagrangian</u> with respect to Blanchfield linking form:

$$Bl_K : A_0(K) \times A_0(K) \longrightarrow \mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}]$$

i.e. $Bl(x, y) = 0 \quad \forall\, x, y \in \operatorname{Ker} i_*$

Hence just always choose η s.t. $Bl_R(\eta,\eta)\neq 0$. Then $\eta\notin \ker i_*$

Higher-Order Alexander modules

$\pi_1(M_K)\to\Gamma$

$\pi_1(W)\to\Gamma$

$$A_\Gamma(M_K)\equiv H_1(M_K;\mathbb{Z}\Gamma)$$

$$A_\Gamma(W)=H_1(W;\mathbb{Z}\Gamma)$$

and $Bl_\Gamma: A_\Gamma(M_K)\times A_\Gamma(M_K)\longrightarrow \mathcal{K}\Gamma/\mathbb{Z}\Gamma$

Second Fundamental Theorem of COT

$\mathrm{Ker}\left(A_\Gamma(M_k) \longrightarrow A_\Gamma(W) \right)$ is <u>isotropic</u> with respect to $\mathcal{B}\ell_\Gamma$ if

W is n-solution and

$$\pi_1(W) \longrightarrow \pi_1(W) / \pi_1(W)^{(n)} \longrightarrow \underset{\text{PTFA}}{\Gamma}$$

Final Principal: for iterated satellite knots

$$K = R_n \cdots \circ R_2 \circ R_1(J)$$

the ordinary Alexander modules of knots R_i show up "inside" the higher-order Alexander modules of K.