QUANTUM CYCLOTOMIC ORDERS OF 3-MANIFOLDS

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Abstract. This paper provides a topological interpretation for number theoretic properties of quantum invariants of 3-manifolds. In particular, it is shown that the p-adic valuation of the quantum $SO(3)$-invariant of a 3-manifold M, for odd primes p, is bounded below by a linear function of the mod p first betti number of M. Sharper bounds using more delicate topological invariants are given as well.

§0. Introduction

Since the birth of quantum topology in the last decade [Jo] [Wi], one of the fundamental problems facing topologists has been to find topological interpretations for the vast array of quantum invariants that have come to light. One common characteristic among these invariants is their rich number theoretic content, and it has been a special challenge to understand which aspects of this number theory have topological significance.

Of central interest are the quantum invariants of 3-manifolds that arise from the representation theory of classical Lie groups [Wi] [RT2]. Roughly speaking there is a complex valued invariant τ_p^G associated with any suitable Lie group G (the *gauge group*) and integer p (the *coupling constant* or *level*). Typically τ_p^G takes values in a cyclotomic field determined by G and p, and this is where the number theory comes into play. There are of course many algebraic invariants in cyclotomic fields. Among the most fundamental are the valuations associated with prime ideals in their rings of integers. The object of this paper is to demonstrate a connection between these valuations and basic topological invariants.

The results obtained here will also be used in the authors' forthcoming paper on a theory of finite type invariants for arbitrary 3-manifolds [CM1]. They provide the basis for the construction of a rich family of new invariants of finite type.

For simplicity we shall limit our investigations to the case when $G = SO(3)$ and p is an odd prime. Similar considerations should apply to the gauge group $SU(2)$, since

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the $SU(2)$ and $SO(3)$ invariants are proportional by a factor involving only classical homotopy theoretic invariants [KM1], and it is expected that results will soon follow for other Lie groups at prime levels. Extending to composite levels may require new ideas.

So consider the quantum $SO(3)$ invariant at an odd prime level p, which will be denoted simply by τ_p. This invariant (normalized as in §4) takes values in the cyclotomic field Q_p of pth roots of unity. Let Λ_p denote the ring of integers in Q_p. There is a distinguished valuation on Q_p, associated with the unique prime ideal H in Λ_p containing p, and its value on any $x \in Q_p$ will be called the *p-order* of x. Now define the *quantum p-order* $\mathfrak{o}_p(M)$ of a 3-manifold M to be the p-order of $\tau_p(M) \in Q_p$. Alternatively $\mathfrak{o}_p(M)$ can be defined as the largest integer m for which $\tau_p(M) \in H^m$; an easy exercise from the definition of τ_p shows that such integers exist.

A beautiful theorem of Hitoshi Murakami [M1] [M2] and Masbaum-Roberts [MR] states that $\mathfrak{o}_p(M) \geq 0$, that is τ_p actually takes values in Λ_p. (Similar results hold for many of the other classical Lie groups by recent work of Takata [Ta] and Masbaum-Wenzl [MW].) It was also shown by Murakami [M2] that $\mathfrak{o}_p(M) = 0$ if and only if M is a $\mathbb{Z}_p$-homology sphere. This was the first indication that topological information might be carried by $\mathfrak{o}_p(M)$.

In this paper it will be shown that the quantum p-order of M is bounded below by a linear function of its mod p first betti number. In particular

$$\mathfrak{o}_p(M) \geq b_p(M)\, n/3$$

where $b_p(M) = \mathrm{rk}(H_1(M; \mathbb{Z}_p))$ and $n = (p-3)/2$. Furthermore, it will be seen that this is the best possible betti number bound for the order when $b_p \equiv 0 \pmod 3$. For $b_p = 1$ or 2, an improved lower bound $\mathfrak{o}_p \geq n$ will be established and shown to be sharp. It is also shown how to strenghthen these bounds by considering a more refined topological invariant, the "Milnor degree" (see §2, Theorem 4.3 and Remark 4.6).

Along the way, a family of $\mathbb{Z}_{p^k}$-valued 3-manifold invariants τ_p^d will be introduced. Using the methods of this paper, it can be shown that these invariants are of "finite type" and converge to τ_p. One striking consequence of this fact is that any two 3-manifolds with unequal quantum $SO(3)$ invariants can be distinguished by finite type invariants (see [CM1] for details, and [R2] for the special case of rational homology spheres).

The paper is organized as follows. In §1 we describe two families of Laurent polynomials, quantum integers and Jones polynomials (including Ohtsuki's version [O1]), and use these to define the p-bracket of a framed link. The p-bracket is the key ingredient in the definition of the quantum invariant τ_p, given in §4 along with the definition of the invariants τ_p^d. The relationship of the Jones and Ohtsuki polynomials with the Kontsevich integral is developed in §2 following the work of Le-Murakami [LM1] and Kricker-Spence [KS]. This leads to a lower bound for the power series order of the Ohtsuki polynomial of a link, which is the essential topological input to the bounds obtained in this paper. In §3 we study the cyclotomic images of the Laurent polynomials introduced in §1, and lower bounds for their p-orders are given which strengthen previous bounds of Murakami [M1] and Ohtsuki [O1]. In §4, these results are combined with a diagonalizing lemma of Murakami and Ohtsuki [M2] to establish bounds on the quantum p-orders of 3-manifolds. Finally in §5 the examples needed to sharpen these bounds are presented.

§1. Laurent polynomials

In this section we introduce two families of Laurent polynomials that arise in the study of quantum invariants, one coming from number theory – *quantum integers* – and the other from topology – *Jones polynomials*. These are combined at the end of the section to form the *p-bracket* of a framed link, which is the key ingredient in the surgery definition of the quantum invariant τ_p given in §4.

The ring Λ and order

Throughout the paper, Λ will denote the ring $\mathbb{Z}[t,t^{-1}]$ of integer Laurent polynomials in an indeterminant t. The variables $s=t^2$ and $q=t^4$ will also be used, as is common in the quantum topology literature. These variables will reappear in §3 as roots of unity.

The following notion, and variations thereof, will be central to our investigations.

Definition 1.1. The *order* $\mathfrak{o}(f)$ of a Laurent polynomial $f\in\Lambda=\mathbb{Z}[t,t^{-1}]$ is the order of $t=1$ as a zero of f.

Thus $\mathfrak{o}(f)$ is the lowest degree appearing in the Taylor expansion of f about 1. The related notion of *p-order*, for any prime p, is introduced in §3. Lower bounds for the orders and p-orders of some of the polynomials introduced in this section will be derived in later sections.

Remark 1.2. An equivalent way to define $\mathfrak{o}(f)$ is by means of the substitution $h=t-1$. This embeds Λ into the ring of formal integer power series in h (by mapping t to $1+h$ and t^{-1} to $(1+h)^{-1}=1-h+h^2-+\cdots$) and then $\mathfrak{o}(f)$ is the smallest power of h with a non-zero coefficient in the series for f.

It should be noted that the variable h has often been used in the literature to stand for $q-1$ or $\log(q)$ rather than $t-1$. Thus Laurent polynomials in t are transformed by substituting $t=(1+h)^{1/4}$ or $t=\exp(h/4)$ into *rational* power series in h. Fortunately the induced order valuations on Λ are equal. Indeed the lowest order terms for any substitution of the form $t=1+ah+O(h^2)$, with $a\neq 0$, will occur in the same degree, and so the order of f is well defined independent of such choices for the variable h. To avoid confusion we shall stick with the assignment $h=t-1$ throughout this paper, using different symbols for other substitutions as the need arises. For example $\hbar=\log(q)$ is used in §2. Conway's substitution $z=s-s^{-1}$ is also useful in simplifying the formulas in §3, and provides the same notion of order since $z=4h+O(h^2)$.

Quantum integers

For each integer k define the *quantum integer* $[k] = (s^k - s^{-k})/(s - s^{-1})$, and more generally the *framed quantum integers*

$$(a, k] = t^{a(k^2-1)}\,[k] \tag{1}$$

for any integer a; note that $(0, k] = [k]$. These elements of Λ are ubiquitous in the theory of quantum invariants. Other versions of the quantum integers also arise frequently, such as the classical *Gauss polynomials* $\langle k\rangle = (t^k - 1)/(t - 1)$, but we shall make little use of them (except briefly in §3).

Observe that $[k]$ can be written as a polynomial in $[2]$. Indeed it follows by induction from the elementary identity $[k] = [2][k-1] - [k-2]$ that

$$[k] = \sum_{j=0}^{k/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1} \tag{2}$$

for $k \geq 0$ (these are renormalized Chebyshev polynomials), and $[-k] = -[k]$. The sum is (by convention) over all *integers* $0 \leq j \leq k/2$, and so the upper limit is actually $k/2 - 1$ for k even, since the binomial coefficient vanishes when $j = k/2$, and $(k-1)/2$ for k odd.

More generally consider the two parameter family of *cabled quantum integers*

$$[k, c) = \sum_{j=0}^{k/2} (-1)^j \binom{k-j-1}{j} \binom{k-2j-1}{c} [2]^{k-2j-1-c} \tag{3}$$

in Λ; note that $[k, 0) = [k]$. These are defined for non-negative integers k and c by (3), and for arbitrary k and c by declaring $[-k, c) = -[k, c)$ and $[k, c) = 0$ for $c < 0$. They arise in conjuction with the framed quantum integers in the formulas below for quantum invariants. In particular, the *p-sums*

$$(a|c) = \sum_{k=1}^{p/2} (a, k][k, c), \tag{4}$$

defined for any odd integer $p \geq 3$, play a special role. Note that the only dependence on p is in the upper limit of summation, which is effectively $(p-1)/2$ since p is odd. Since the value of p is generally fixed, this dependence is not made explicit in the notation.

It will be shown in Proposition 3.7 that if p is prime and a is a multiple of p, then the p-sum $(a|c)$ is divisible by h^{p-3-2c} when viewed as an element of the cyclotomic quotient Λ_p of Λ. This is the main new technical result used here to establish bounds for the quantum p-orders of 3-manifolds.

Jones polynomials

The key topological input in the construction of quantum invariants of 3-manifolds is the *Jones polynomial* V_L of an oriented link L in the 3-sphere [Jo]. The version used here is a Laurent polynomial in the variable $s = t^2$ characterized by $s^2 V_{L_+} - s^{-2} V_{L_-} = (s - s^{-1}) V_{L_0}$ and $V_{\bigcirc} = [2]$. Here L_+, L_-, L_0 is the usual skein triple and $\bigcirc$ is the unknot. By convention $V_\emptyset = 1$, where $\emptyset$ is the empty link.

We shall actually use a variant J_L of V_L, arising naturally in the quantum group approach to the subject [RT1] [KM1], which is independent of the orientation on L. It is defined by

$$J_L = s^{3\lambda_L} V_L \tag{5}$$

where λ_L denotes the sum of all the pairwise linking numbers of L. Equivalently $J_L = (-1)^\ell K_L$, where ℓ is the number of components in L and K_L is the Kauffman bracket of any zero-framed diagram for L, normalized to be 1 on the empty link [Ka]. Clearly $J_{\bigcirc} = [2]$, and more generally $J_{\bigcirc^c} = [2]^c$ where $\bigcirc^c$ denotes the unlink of c components, since J is multiplicative under distant unions $\sqcup$ of links. (This polynomial is the zero-framed version of the invariant J_L defined in §4 of [KM1]; the framed version has value $(a, 2]$ on the a-framed unknot.)

Using a cabling operation one may extend the definition of J to the class of *colored* links (L, k) in S^3. The *coloring* $k = (k_1, \ldots, k_\ell)$ is a list of positive integers assigned to the components of L (representing the dimensions of simple modules in the quantum group approach). The *colored Jones polynomial* of (L, k) is then defined by the following formula, written in the multi-index notation of [KM1]

$$J_{L,k} = \sum_{j=0}^{k/2} (-1)^j \binom{k-j-1}{j} J_{L^{k-2j-1}}, \tag{6}$$

(This is the zero-framed version of the invariant $J_{L,\mathbf{k}}$ in [KM1].) Here L^c denotes the *c-cable* of L, where the *cabling* $c = (c_1 \cdots, c_\ell)$ is a list of integers, obtained by replacing the ith component of L by c_i parallel copies of itself with pairwise linking numbers equal to zero; by convention $J_{L^c} = 0$ if any $c_i < 0$. The multi-index notation in (6) is to be interpreted as follows: the sum is over all lists $j = (j_1, \cdots, j_\ell)$ with $0 \le j_i < k_i/2$, and the signs and binomial coefficients are products of the corresponding terms for each i. Note that $J_{L,k}$ can be defined for arbitrary integer colorings by requiring it to be an odd function of any given color.

The reader may have noticed a similarity between equations (2) and (6). Indeed, setting $k = 2$ in (6) gives $J_{L,2} = J_L$, and so (6) says that $J_{L,k}$ is the kth Chebyshev polynomial in $J_{L,2}$ with cables replacing powers. In particular, taking the unknot for L shows that $J_{\bigcirc,k} = [k]$. (The framed version of this invariant has value $(a, k]$ on the k colored unknot with framing a.)

Ohtsuki polynomials

Finally, we introduce another version ϕ_L of the Jones polynomial due to Tomotada Ohtsuki [O1]. First consider the free $\mathbb{Z}$-module $\mathcal{L}$ with basis consisting of all oriented links in S^3 (the orientation is only relevant to the discussion of the Kontsevich integral below). Note that the invariant J can be extended uniquely to a linear functional $J : \mathcal{L} \to \Lambda$.

Next consider the projection $\pi : \mathcal{L} \to \mathcal{L}$ defined by

$$\pi(L) = \sum_{S<L} (-1)^{\ell-s}\, S|L \tag{7}$$

where $S|L = S \sqcup \bigcirc^{\ell-s}$, obtained from L by replacing each component not in S with a distant unknot. The sum is over all sublinks S of L, including the empty link, and $\ell = \#L$, $s = \#S$. That π is a projection follows readily from the fact that $\pi(S \sqcup T) = \pi(S) \sqcup \pi(T)$ and $\pi(\bigcirc) = 0$.

Now define ϕ to be the composition $J\pi$, that is

$$\phi_L = J_{\pi(L)} = \sum_{S<L} [-2]^{\ell-s} J_S. \tag{8}$$

Evidently ϕ_L is an integer Laurent polynomial in $s = t^2$, and in particular an element of Λ. It will be called the *Ohtsuki polynomial* of L (see Remark 1.3 below).

Observe that J can be expressed in terms of ϕ as follows

$$J_L = \sum_{S<L} [2]^{\ell-s}\, \phi_S. \tag{9}$$

To see this, note that any link L can be recovered from the projections of its sublinks as $L = \sum_{S<L} \pi(S) \sqcup \bigcirc^{\ell-s}$. Indeed the right hand side is $\sum_{R<S<L} (-1)^{s-r}\, R|L$ by definition, and this can be rewritten as $\sum_{R<L} \big(\sum_{k=0}^{\ell-r} (-1)^k \binom{\ell-r}{k}\big)\, R|L$, which equals L since the inner sum vanishes for $R \neq L$. Equation (9) follows.

More generally, the colored Jones polynomial $J_{L,k}$ can be rewritten as a linear combination of ϕ_{L^c} for cables $c < k$ (i.e. $c_i < k_i$ for each i), with coefficients the multi-index versions $[k, c) = \prod [k_i, c_i)$ of the cabled quantum integers defined in (3):

Lemma 1.3. $J_{L,k} = \displaystyle\sum_{c=0}^{k-1} [k, c)\, \phi_{L^c}$

Proof. First use (9) to replace $J_{L^{k-2j-1}}$ in the definition (6) of $J_{L,k}$ with the sum of $[2]^{k-2j-1-s} \phi_S$ over all sublinks S of the cabling L^{k-2j-1} of L. Now each S is again a cabling L^c of L, appearing $\binom{k-2j-1}{c}$ times in the sum. Collecting terms yields the result. $\square$

Remark 1.4. The polynomial ϕ_L defined here differs from Ohtsuki's original version Φ_L by a factor of $[2]^\ell$. Indeed by definition $\Phi_L = (-1)^\ell X_{\delta(L)}$, where $X_S = J_S/[2]^s$ is Hitoshi Murakami's normalization of J_S [M1] and δ is the involution on $\mathcal{L}$ given by

$$\delta(L) = \sum_{S<L} (-1)^s S.$$

(That δ is an involution follows by an argument analogous to the derivation of (9) above.) Thus $\Phi_L = \phi_L/[2]^\ell$. One advantage of ϕ over Φ is that it is an honest Laurent polynomial, thus justifying its name, taking values in Λ rather that Λ localized at [2]. Ohtsuki's normalization on the other hand leads to a pleasing symmetry between X and Φ, namely $X_L = (-1)^\ell \Phi_{\delta(L)}$. This is immediate from the fact that δ is an involution, and yields an alternative derivation of (9).

The p-bracket

Fix an odd integer $p \geq 3$. The key ingredient in the surgery definition of the quantum invariant τ_p is the *p-bracket* of a *framed* link L in the 3-sphere. It is an integer Laurent polynomial in t, that is an element of Λ, defined by

$$\langle L \rangle = \sum_{k=1}^{p/2} (a,k]\, J_{L,k} \tag{10}$$

where $a = (a_1, \dots, a_\ell)$ is the list of integer framings on L. (As with the p-sums defined in (4), the dependence on p is not explicit in the notation, but is to be understood from the context.) Multi-index notation is being used as usual: the sum is over all colorings $k = (k_1, \dots, k_\ell)$ of L with $1 \leq k_i < p/2$ for each i, and the coefficients $(a,k]$ are multi-index versions $\prod (a_i, k_i]$ of the framed quantum integers defined in (1).

For any integer a, set $b_a = \langle \bigcirc_a \rangle$, the p-bracket of the a-framed unknot. This coincides with the associated zero-cabled p-sum defined in (4)

$$b_a = (a|0) \tag{11}$$

since $J_{\bigcirc,k} = [k]$.

We conclude this section with an expression for the p-bracket of L in terms of Ohtsuki polynomials of cablings L^c of L and the associated multi-index versions $(a|c) = \prod (a_i|c_i)$ of the p-sums defined in (4).

Proposition 1.5. *The p-bracket of a framed link L in the 3-sphere with framings a can be written as $\langle L \rangle = \sum_{c=0}^{n} (a|c)\, \phi_{L^c}$ where $n = (p-3)/2$.*

Proof. By Lemma 1.3 and the definition of the p-bracket,

$$\langle L \rangle = \sum_{k=1}^{p/2} \sum_{c=0}^{k-1} (a,k][k,c)\, \phi_{L^c}.$$

Since $[k,c) = 0$ for $c \geq k$, the upper limit of the inner sum can be replaced with n. Switching the order of summation then gives the result. $\square$

§2. Relations with the Kontsevich integral

The Jones and Ohtsuki polynomials of a link in the 3-sphere can both be interpreted in terms of the Konsevich integral of the link. For the Jones polynomial, this was first made explicit in the work of Le and Murakami [LM1], motivated by earlier work of Drinfeld on quasi-Hopf algebras [Dr]. For the Ohtsuki polynomial, this was elucidated in a recent paper of Kricker and Spence [KS], following ideas suggested in the seminal work of Le on finite type invariants of homology spheres [Le]. This interpretation of the Ohtsuki polynomial can be made particularly transparent using the projection $\pi : \mathcal{L} \to \mathcal{L}$ defined in (7) , as explained below.

The Kontsevich integral

Recall the *Kontsevich integral* $\hat{Z}_L$ of a *framed* oriented link L in S^3, normalized as in [LMO]. It is an element of the completion $\mathcal{A}(L)$ (with respect to degree) of the rational vector space generated by Feynman diagrams on L modulo the appropriate relations (AS, IHX and STU). Here a *Feynman diagram* on L consists of an abstract vertex-oriented uni-trivalent graph with all of its univalent vertices on L (see [LMO] for details). This graph is generally referred to as the *dashed graph* of the diagram, as it is drawn with dashed lines in pictures. The univalent vertices are called *external* vertices, and the trivalent ones are called *internal*. The *degree* of the diagram is half the total number of vertices.

Any element in $\mathcal{A}(L)$ can be expanded (in many ways) as an *infinite* linear combination of diagrams; such an expansion will be called a *Feynman series* for the element. A simply connected component of the dashed graph of any diagram in the series will be called a *tree*, and if it has all but possibly one external vertex lying on a single component of L, then it will be called a *thin tree*. These notions will be used later in this section.

For the present purposes we shall only consider *zero-framed* links, and shall also assume that all links come equipped with an ordering for their components so that $\mathcal{A}(L)$ is identified with $\mathcal{A}(\bigcirc^\ell)$ for any ℓ-component link L. The Kontsevich integral can then be viewed as a $\mathbb{Z}$-linear map $\hat{Z} : \mathcal{L} \to \mathcal{A}$ where where $\mathcal{L}$ is the free $\mathbb{Z}$-module generated by oriented links in S^3 and $\mathcal{A} = \oplus_{\ell=0}^{\infty} \mathcal{A}(\bigcirc^\ell)$. This map is a morphism for distant unions, i.e. $\hat{Z}_{J \sqcup K} = \hat{Z}_J \sqcup \hat{Z}_K$. As is standard practice, ν will denote the value of $\hat{Z}$ on the unknot, which is a unit in $\mathcal{A}(\bigcirc)$ with respect to the connected sum operation $\#$ (see §1.1 in [LMO]).

In fact it is convenient to use a related invariant $Z : \mathcal{L} \to \mathcal{A}$ defined by

$$Z_L = \hat{Z}_L \,\#\, (\nu^{-1})^\ell \tag{12}$$

meaning that one takes the connected sum of a copy of ν^{-1} with each component of L in each term of $\hat{Z}_L$. (This is the zero-framed version of the invariant $\hat{Z}'$ of [KS].) Note that Z is also a morphism for distant unions.

Connections with the Jones and Ohtsuki polynomials

To interpret the Jones and Ohtsuki polynomials (which take values in the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$) in terms of the Kontsevich integral, it is useful to expand these polynomials as power series in $\hbar = \log(q)$ where $q = t^4$ (see Remark 1.2). In particular, consider the embedding $E : \Lambda \to \mathbb{Q}[[\hbar]]$ which maps t to $\exp(\hbar/4)$. For any function F mapping into Λ, write $\hat{F}$ for the composition EF. Thus for example $\hat{J}_L$ is the power series obtained from the Jones polynomial J_L by substituting $\exp(\hbar)$ for q, and $\hat{\phi}_L = \hat{J}_{\pi(L)}$ is the series for the corresponding Ohtsuki polynomial.

Now following [KS], let $\Gamma : \mathcal{A} \to \mathbb{Q}[[\hbar]]$ denote the weight system associated with the trace form in the fundamental representation of $sl(2, \mathbb{C})$. In fact, it is convenient to use a renormalization W of Γ defined by $W(D) = (\widehat{[2]}/2)^\ell\, \Gamma(D)$ on any Feynman diagram D on an ℓ component link. Note that W preserves order (the lowest degree in a power series) since Γ preserves degree and $\widehat{[2]} = 2 + \hbar^2/4 + \cdots$ is of order zero. Also let $P : \mathcal{A} \to \mathcal{A}$ denote the projection which sends a diagram D on L to itself if it has vertices on *every* component of L, and to zero otherwise. Then the interpretation of the Jones and Ohtsuki polynomials in terms of the Kontsevich integral can be expressed as follows.

Lemma 2.1. *The diagram*

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\pi} & \mathcal{L} & \xrightarrow{J} & \Lambda \\ {\scriptstyle Z}\downarrow & & {\scriptstyle Z}\downarrow & & \downarrow{\scriptstyle E} \\ \mathcal{A} & \xrightarrow[P]{} & \mathcal{A} & \xrightarrow[W]{} & \mathbb{Q}[[\hbar]] \end{array}$$

commutes, and so (a) (Le-Murakami) $\hat{J} = WZ$, *and* (b) (Kricker-Spence) $\hat{\phi} = WPZ$.

Proof. The commutativity of the square on the right, which is equivalent to (a), is the result of Le-Murakami [LM1] mentioned above. Actually they show $EJ = \Gamma\hat{Z}$, but it is easy to verify that $\Gamma\hat{Z} = WZ$ using the identities $\Gamma(D\#D') = \frac{1}{2}\Gamma(D)\Gamma(D')$, $\Gamma(\bigcirc) = 2$ and $\Gamma(\nu) = \widehat{[2]}$ (see [LM2][KS]).

For the square on the left, observe that for any sublink S of a link $L \in \mathcal{L}$, the Kontsevich integral $Z_{S|L}$ of $S|L = S \sqcup \bigcirc^{\ell - s}$, which appears in the definition of π in (7), is the sum of the terms in Z_L whose diagrams have all their vertices on S. This is an elementary consequence of the work of Le and Murakami which describes how $\hat{Z}_S$ and $\hat{Z}_L$ are related (see Proposition 1.1 in [LMO]). Now the commutativity of the left-hand square follows by the inclusion-exclusion principle of combinatorics. This implies the commutativity of the outermost rectangle (shown directly in [KS]) which is equivalent to (b). $\square$

Orders of Ohtsuki polynomials

Kricker and Spence's formula $\hat{\phi}_L = W(P(Z_L))$ led them to a striking lower bound for the order of the Ohtsuki polynomial of any link L whose pairwise linking numbers all vanish. Such a link will be called a *diagonal* (or *algebraically split*) link.

Theorem 2.2. (Kricker-Spence [KS,§3]) *Let L be a diagonal link with ℓ components. Then the order $\mathfrak{o}(\phi_L)$ of the Ohtsuki polynomial of L is greater than or equal to $4\ell/3$.*

Using different methods, Ohtsuki had previously obtained a bound in terms of the *maximum cabling index* of the link, which is by definition the maximum number m of mutually parallel, algebraically unlinked components of the link. Although weaker than the Kricker-Spence bound in many cases, Ohtsuki's bound is stronger whenever $m > \ell/3$, a situation that will arise in §4 in establishing the p-order bounds for 3-manifolds of small betti number.

Theorem 2.3. (Ohtsuki [O1,§3.4]) *Let L be a diagonal link with ℓ components and maximum cabling index m. Then the order $\mathfrak{o}(\phi_L)$ of the Ohtsuki polynomial of L is greater than or equal to $\ell + m$.*

The theorem of Kricker and Spence follows from their formula by an easy counting argument, after making some elementary observations about the Kontsevich integral of a diagonal link. In fact a recent result of Habegger and Masbaum [HM] (motivated by earlier work of Rozansky [R1] and Le [Le]) shows how to generalize these observations to links with vanishing higher order linking numbers, and so we present the proof in this more general context. Since Ohtsuki's theorem can also be viewed in this context, we formulate a theorem below that includes both results.

First define the *Milnor degree* of a link L in S^3 to be the degree ($=$ length-1) of the first nonvanishing $\bar{\mu}$-invariant of L (see [Mi]). Thus *every* link has degree ≥ 1, while the diagonal links are those of degree ≥ 2. If all the $\bar{\mu}$-invariants of L vanish (e.g. if L is a knot, a boundary link or more generally the fusion of a boundary link [Co]) then L is said to have infinite Milnor degree.

Now the result of Habegger and Masbaum (§6.10 in [HM]) is that the Kontsevich integral $\hat{Z}_L$ of a link L of Milnor degree d has a Feynman series in which every tree has degree at least d. Of course this is also true for the normalization Z_L, since $\nu = \hat{Z}_{\bigcirc}$ can be written as a linear combination of treeless diagrams (by the same result). In fact the proof in [HM] shows more: One can arrange that for each sublink S of L, every tree with all of its external vertices on S has degree no less than the Milnor degree of S; thus for example there will be no chords between any pair of components with linking number zero, even if L is not diagonal. Furthermore, thin trees (ones with all but possibly one vertex on a single component of L) can be avoided in the series if L is diagonal. A series satisfying these conditions will be called a *Milnor series* for Z_L.

Lemma 2.4. *The Kontsevich integral Z_L of any link L has a Milnor series.*

Proof. We assume that the reader is familiar with the arguments presented in §6 of [HM], and adopt the terminology and notation used there.

Choose a string link λ representing L. By the proof of §6.9–10 in [HM], it suffices to show that Z^t_λ (which is an element of $\mathcal{A}^t(\ell)$) has a Milnor series. Now the key observation, as in [HM], is that Z^t_λ is *group like*, and so can be expressed as the exponential of some primitive ξ which is a linear combination of tree diagrams. For each string sublink $\sigma < \lambda$, let ξ_σ denote the sum of the diagrams in ξ which have vertices on every component of σ, but on no other components of λ. Thus $\xi = \sum_{\sigma<\lambda} \xi_\sigma$.

As in the proof of Lemma 2.1 above, it follows from a formula of Le and Murakami (Proposition 1.1.3 in [LMO]) that $Z^t_{\sigma|\lambda} = \exp(\sum_{\tau<\sigma} \xi_\tau)$, and so by Theorem 6.2 in [HM], no tree in ξ_σ is of degree less that the Milnor degree of σ. In particular if L is diagonal, then there are no chords, and any other thin trees can be omitted as well since they vanish in $\mathcal{A}^t(\ell)$. Therefore $\exp(\xi)$ is a Milnor series for Z^t_λ. $\square$

Theorem 2.5. *Let L be an ℓ-component link of Milnor degree d. Then the order $\mathfrak{o}(\phi_L)$ of the Ohtsuki polynomial of L is greater than or equal to $2\ell d/(d+1)$. Furthermore, if L is diagonal ($d \geq 2$) with maximum cabling index m, then $\mathfrak{o}(\phi_L) \geq \ell + m$.*

In particular every link satisfies $\mathfrak{o}(\phi_L) \geq \ell$. For diagonal links the bounds can be incorporated into a single formula

$$\mathfrak{o}(\phi_L) \geq \ell + \max(\frac{d-1}{d+1}\ell, m),$$

and for links of infinite Milnor degree $\mathfrak{o}(\phi_L) \geq 2\ell$.

Proof of (2.5). By Lemma 2.1 and the fact that W preserves order, it suffices to show that $P(Z_L)$ has a Feynman series consisting of diagrams of degree $\geq 2\ell d/(d+1)$, and $\geq \ell + m$ when L is diagonal.

First observe that the degree d_D of a diagram D can be computed as the difference $x_D - e_D$, where x_D is the number of external vertices in D and e_D is the euler characteristic of its dashed graph, or alternatively as the sum of the *weights* of the external vertices of D. Here the weight of a vertex is defined as the ratio d_C/x_C, where C is the *component* of D containing the vertex. By the first formula for the degree, this weight is $d/(d+1)$ if C is a tree of degree d, and is ≥ 1 otherwise.

It follows from Lemma 2.4 that Z_L has a Feynman series with all external vertices of weight $\geq d/(d+1)$. The corresponding series for $P(Z_L)$ is obtained by eliminating certain diagrams, leaving only those with at least one vertex on each component of L. Moreover, diagrams with *exactly* one vertex on some component vanish in $\mathcal{A}$ by the STU relation. The remaining diagrams have at least 2ℓ external vertices, and therefore degree $\geq 2\ell d/(d+1)$.

Now consider the case when L is diagonal. By hypothesis $L = J \cup K^m$ where J has $j = \ell - m$ components and K is a knot. Applying Lemma 2.4 again, choose a series for $P(Z_{J\cup K})$ consisting of diagrams with no thin trees, no trees of degree $< d$, and with at least two vertices on each component of $J \cup K$. Cabling this series (following [LM2,§4.1]) produces a similar series for $P(Z_L)$ with the additional property that every tree has at least two vertices on J. Now any diagram D in this series has degree $d_D = x_D - e_D \geq x_D - t_D$, where t_D denotes the number of trees in D. Since $x_D = 2\ell + k$ for some $k \geq 0$, there are at most $2j+k$ vertices on J, and so $t_D \leq j + k/2$. Therefore $d_D \geq 2\ell + k - (j + k/2) \geq \ell + m$. □

§3. Cyclotomacy

In this section we introduce *cyclotomic orders* in the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$ of Laurent polynomials. In particular, for each prime p we define the *p-order* on Λ in terms of a certain distinguished valuation $\mathfrak{o}_p$ on the quotient ring Λ_p by the ideal generated by the pth cyclotomic polynomial. Bounds will then be established for the p-orders of the polynomials defined in §1, namely the p-sums (Propositions 3.6 and 3.7), and the Ohtsuki polynomials and p-brackets of links (Proposition 3.5 and Theorem 3.10).

The ring Λ_p of cyclotomic integers

Fix an odd prime $p = 2n + 3$. The cyclotomic polynomial $\varphi_p = (t^p - 1)/(t - 1)$ generates a prime ideal in $\Lambda = \mathbb{Z}[t, t^{-1}]$. Let Λ_p denote the quotient of Λ by this ideal,

$$\Lambda_p = \Lambda/(\varphi_p),$$

and Q_p denote the field of fractions of Λ_p. Of course Q_p can be identified with the cyclotomic field of *complex* pth roots of unity, and Λ_p with its ring of integers. Indeed for any primitive complex pth root ζ, there is a unique ring homomorphism $\Lambda \to \mathbb{C}$ mapping t to ζ, and this induces an isomorphism $Q_p \cong \mathbb{Q}(\zeta)$ carrying Λ_p to $\mathbb{Z}[\zeta]$. Various properties of Λ_p can be deduced from this observation. For example the classical fact that the ring of integers in a number field is a Dedekind domain shows that ideals in Λ_p factor uniquely into prime ideals.

For convenience we retain the symbol t for the image of t in Λ_p, and continue to use the notation $s = t^2$ and $q = t^4$. Thus a Laurent polynomial in t may be used to represent either an element in Λ or an element in Λ_p, depending upon the context. Note however that these two elements may have very different number theoretic properties. For example t, s and q become pth roots of unity, and thus units, in Λ_p. The quantum integers $[k] = (s^k - s^{-k})/(s - s^{-1})$ are also units in Λ_p if k is prime to p, with $[k]^{-1} = [k\bar{k}]/[k]$ (clearly an integral polynomial in t) where $\bar{k}$ is any mod p inverse of k. Similarly the Gauss polynomials $\langle k \rangle = (t^k - 1)/(t - 1)$ are units in Λ_p for k prime to p. Note that $[k]$ is an odd function of k while $\langle k \rangle$ is not. Both are periodic of period p.

The prime ideal H and p-order in Λ_p

The polynomial $h = t - 1$, which is evidently a prime in Λ, remains prime as an element of Λ_p (see for example §1.4 in [Wa]). In contrast p is not a prime in Λ_p. It is in fact a power of h times a unit, as seen by the calculation $p = \varphi_p(1) = (t-1)\cdots(t^{p-1}-1) = \langle p-1\rangle!\, h^{p-1}$. (Here $\langle p-1\rangle!$ is the product $\langle 1\rangle\cdots\langle p-1\rangle$ of Gauss polynomials.) Thus h generates a prime ideal H in Λ_p and

$$P = H^{p-1}$$

where $P = (p)$ is the ideal generated by p. This is an instance of the prime factorization of ideals in Λ_p. The uniqueness of this factorization shows that H is the only prime ideal in Λ_p containing p.

The powers of H form a descending sequence

$$\Lambda_p = H^0 \supset H^1 \supset H^2 \supset \cdots$$

of ideals with trivial intersection. Additively, each of these is free abelian of rank $p-1$. Indeed the consecutive powers $h^k, \cdots, h^{k+p-2}$ form a basis for H^k as a $\mathbb{Z}$-module. Thus any non-zero element x in Λ_p lies in a unique smallest H^{k_0}, and can be written uniquely as an integer polynomial $x_k = \sum_{d=0}^{p-2} x_{k,d} h^{k+d}$ in h for each $k \le k_0$. We will call x_{k_0} the *normal form* of x, and x_0 (which is of degree $\le p-2$) the *reduced form* of x.

The integer k_0, which is the *order* of the normal form of x in the sense of Remark 1.2, is called the *p-order* of x, and will be denoted by $\mathfrak{o}_p(x)$. In simple terms $\mathfrak{o}_p(x)$ is the exponent of the highest power of h that divides x (as an element of Λ_p). This notion can be lifted to the ring Λ by declaring the p-order of a Laurent polynomial f, also written $\mathfrak{o}_p(f)$, to be the p-order of its image in Λ_p. These notions will be recast below in the context of valuation theory.

Remarks 3.1. (a) In discussing p-order, h may be replaced by any other generator of H, that is any associate of h in Λ_p. Among the possible choices are $q-1$ and $z = s - s^{-1}$ (cf. Remark 1.2), or more generally any of the elements $h_{j,k} = t^j - t^k$ for $j \not\equiv k \pmod p$, since $h_{j,k} = (\langle j–k\rangle t^k)\, h$. Furthermore some formulas are more revealing with a different choice of generator. For example the formula $p = \langle p-1\rangle! h^{p-1}$ above has an analogue using the generator z, namely $p = [p-1]! z^{p-1}$. This is proved in the same way, but starting with the factorization $p = (q-1)\cdots(q^{p-1}-1)$. But now exploiting the fact that $[k]$ is an *odd* periodic function of k, this can be rewritten as $p = (-1)^m([m]!z^m)^2$ where $m = (p-1)/2$. This recovers the well known fact that $(-1)^m p$ is a square in Λ_p. Moreover, its square roots can be identified with the Gauss sums

$$G_a \underset{\text{def}}{=} \sum_{k=1}^{p} t^{ak^2} = \left(\frac{a}{p}\right)(-1)^m[m]!z^m$$

by an easy argument using the classical formula $\prod_{k=1}^m (t^{2k-1} - t^{-(2k-1)})$ for $G = G_1$ [IR,§6.4.4]. Here $(-)$ is the Legendre symbol.

(b) An integral polynomial $\sum_{d=0}^{p-2} x_d h^{k+d}$ in h with $x_0 \neq 0$ is in normal form if and only if x_0 is prime to p. For if p divides x_0, then the first term $x_0 h^k$ can be killed by substracting a multiple of the cyclotomic polynomial φ_p, since φ_p has constant coefficient p when written as a polynomial in h; indeed $\varphi_p = (t^p - 1)/(t-1) = ((h+1)^p - 1)/h$, and so

$$\varphi_p = p + \sum_{k=2}^{p-1} \binom{p}{k} h^{k-1} + h^{p-1} \tag{13}$$

by the binomial theorem. The converse follows from the fact that p is the only prime (rational) integer divisible by h.

(c) If an element of Λ_p is given as the image of a Laurent polynomial in t, then it can be put in either normal or reduced form in the following way: First rewrite the polynomial using the relation $t^p = 1$ as an *honest* polynomial in t, i.e. an element in $\mathbb{Z}[t]$. Then substitute $t = 1 + h$ to obtain an element in $\mathbb{Z}[h]$. Finally subtract a suitable multiple of φ_p to put this in the required form, working from the "bottom-up" for normal form as indicated in the previous remark, and from the "top-down" for reduced form. For example if $p = 3$, then working mod $\varphi_3 = 3 + 3h + h^2$ we have $t + 2t^{-1} \equiv t + 2t^2 \equiv 3 + 5h + 2h^2 \equiv 2h + h^2$ (normal form) or $-3 - 2h$ (reduced form). Thus $\mathfrak{o}_3(t + 2t^{-1}) = \mathfrak{o}_3(2h + h^2) = 1$.

Valuation theory

A *prevaluation* on an integral domain D is a nonconstant map $\mathrm{v} : D \to \mathbb{Z} \cup \{\infty\}$ satisfying

$$\begin{aligned} &\text{(a)} \quad \mathrm{v}(xy) = \mathrm{v}(x) + \mathrm{v}(y) \\ &\text{(b)} \quad \mathrm{v}(x+y) \geq \min(\mathrm{v}(x), \mathrm{v}(y)). \end{aligned}$$

It follows from (a) that $\mathrm{v}(1) = 0$, and so $\mathrm{v}(u^{-1}) = -\mathrm{v}(u)$ for any unit u in D. In particular $\mathrm{v}(x/y) = \mathrm{v}(x) - \mathrm{v}(y)$ if D is a field. It also follows from (a) that $\mathrm{v}(0) = \infty$.

The set $\mathcal{R}_\mathrm{v}$ of elements x in D with $\mathrm{v}(x) = \infty$ is called the *radical* of v, and is evidently an ideal in D. If $\mathcal{R}_\mathrm{v} = 0$, then v is called a *valuation* (and so any prevaluation on a field is *a fortiori* a valuation). The set $\mathcal{O}_\mathrm{v}$ of elements with $\mathrm{v}(x) \geq 0$ is a subring of D called the *(pre)valuation ring* of v. If $\mathcal{O}_\mathrm{v} = D$ then v is said to be *positive*. More generally consider the descending filtration $\cdots \supset \mathcal{O}_\mathrm{v}^d \supset \mathcal{O}_\mathrm{v}^{d+1} \supset \cdots$ of subrings of D, where $\mathcal{O}_\mathrm{v}^d = \{x \mid \mathrm{v}(x) \geq d\}$. In particular $\mathcal{O}_\mathrm{v}^\infty = \mathcal{R}_\mathrm{v}$ and $\mathcal{O}_\mathrm{v}^0 = \mathcal{O}_\mathrm{v}$. These are all ideals when v is positive. In general, the value $\mathrm{v}(x)$ is determined by the position of x in this filtration; $\mathcal{O}_\mathrm{v}^d - \mathcal{O}_\mathrm{v}^{d+1}$ is the set of elements with value exactly d.

The standard example of a valuation in elementary number theory is the *p-adic valuation* v_p on $\mathbb{Q}$ which assigns to each rational number x the exponent of p in the prime decomposition of x. It restricts to a positive valuation on the integers. The p-adic valuation has a natural generalization to the cyclotomic field Q_p:

The p-adic valuation and p-order in Q_p

The *p-adic valuation* $\mathfrak{o}_p$ on Q_p assigns to each element x the exponent of the prime ideal H, the unique ideal in Λ_p lying over p, in the prime decomposition of the fractional ideal generated by x. (This is an instance of the general construction of valuations on the field of fractions of a Dedekind domain D from prime ideals in D [BS].) The integer $\mathfrak{o}_p(x)$ will be called the *p-order* of x. This clearly coincides with the definition given above for $x \in \Lambda_p$, since $\mathcal{O}^d_{\mathfrak{o}_p} = H^d$, and so in this case $\mathfrak{o}_p(x)$ can be computed as the order of the normal form of x (which allows for computation in general since $\mathfrak{o}_p(x/y) = \mathfrak{o}_p(x) - \mathfrak{o}_p(y)$). Alternatively, the p-order of $x \in \Lambda_p$ can be computed from the reduced form $\sum_{d=0}^{p-2} x_d h^d$ of x by $\mathfrak{o}_p(x) = \min_d(\mathfrak{o}_p(x_d h^d)) = \min_d((p-1)\mathrm{v}_p(x_d)+d)$ (the first equality holds since the terms in the sum have *distinct* p-orders).

Remark 3.2. The last computational scheme suggests introducing the $\mathbb{Z}$-linear projections

$$\pi^d : \Lambda_p \to \mathbb{Z}_{p^k}, \quad x \to x_{\bar{d}} \ (\mathrm{mod}\ p^k)$$

for $d \geq 0$. Here $x_{\bar{d}}$ is the coefficient of $h^{\bar{d}}$ in the reduced form of x and $k = 1+\lfloor d/(p-1)\rfloor$, where $\bar{d}$ is the least residue of d mod $(p-1)$ and $\lfloor\ \rfloor$ is the greatest integer function. Now $\mathfrak{o}_p(x)$ can be defined as the smallest integer d for which $\pi^d(x) \neq 0$. The motivation for this point of view comes from the study of the "p-adic completion" $\hat{\Lambda}_p = \varprojlim \Lambda_p/P^k$ of the ring Λ_p. Indeed, the π^d's can be grouped in strings of length $p-1$ to give projections $\pi_k : \Lambda_p \to \Lambda_p/P^k \cong (\mathbb{Z}_{p^k})^{p-1}$ which show $\hat{\Lambda}_p$ isomorphic to a direct sum of $p-1$ copies of the p-adic integers. Note that Λ_p embeds in $\hat{\Lambda}_p$, i.e. any element x in Λ_p can be recovered from its projections $\pi^d(x)$ for $d = 0, 1, 2, \ldots$.

p-order in Λ

The restriction of the p-adic valuation to Λ_p induces a positive *pre*valuation on Λ, $\mathfrak{o}_p : \Lambda \to \mathbb{Z}\cup\{\infty\}$, by composing with the natural projection $\Lambda \to \Lambda_p$. This will be called the *p-adic prevaluation* on Λ. It has radical (φ_p), and more generally $\mathcal{O}^d_{\mathfrak{o}_p} = (h^d, \varphi_p)$ where $h^\infty = 0$ by convention. The value $\mathfrak{o}_p(f)$ will be referred to as the *p-order* of f. In other words:

Definition 3.3. The *p-order* $\mathfrak{o}_p(f)$ of a Laurent polynomial $f \in \Lambda$ is the p-order of the image of f in Λ_p.

Note that $\mathfrak{o}_p(f)$ depends only on the equivalence class of f (mod φ_p), and so its computation is facilitated by an appropriate choice of representative. Typically one uses either the normal or reduced form of f, viewed as an element of Λ_p (see Remark 3.1 and the computational schemes for p-order in Λ_p discussed above). One can, however, glean some information about $\mathfrak{o}_p(f)$ without reducing f (i.e. avoiding the last step of the process described in Remark 3.1c). In particular, lower bounds on $\mathfrak{o}_p(f)$ can be obtained by comparing $\mathfrak{o}_p$ with two other naturally defined (pre)valuations on Λ, which we discuss next.

Order and mod p - order in Λ

The *order valuation* $\mathfrak{o}$ on the power series ring $\mathbb{Z}[[h]]$, defined by

$$\mathfrak{o}(\sum x_d h^d) = \min\{d \,|\, x_d \neq 0\},$$

induces a positive valuation on Λ by means of the usual embedding $t \mapsto 1 + h$. We call $\mathfrak{o}(f)$ the *order* of f (cf. Definition 1.1). Similarly define the prevaluation $\mathfrak{o}_{/p}$ on $\mathbb{Z}[[h]]$ by

$$\mathfrak{o}_{/p}(\sum x_d h^d) = \min\{d \,|\, x_d \not\equiv 0 \bmod p\}.$$

This induces a positive prevaluation on Λ with radical (p), and $\mathfrak{o}_{/p}(f)$ is called the *mod p - order* of f. The three prevaluations $\mathfrak{o}$, $\mathfrak{o}_p$ and $\mathfrak{o}_{/p}$ on Λ are related as follows (cf. [M1,§5.5] and [O1,§7.3]).

Lemma 3.4. *Let f be a Laurent polynomial in t and f' denote its derivative with respect to t (or equivalently with respect to $h = t-1$). Then for any integer $d < \mathfrak{o}(f)+p$,*

(a) $\mathfrak{o}_p(f) \geq \mathfrak{o}(f)$, $\mathfrak{o}_{/p}(f) \geq \mathfrak{o}(f)$

(b) $\mathfrak{o}_p(f) \geq d \Longleftrightarrow \mathfrak{o}_{/p}(f) \geq d$

(c) $\mathfrak{o}_p(f) \geq d \implies \mathfrak{o}_p(f') \geq d - 1$.

Note that (a) and (b) say that $\mathfrak{o}_p$ and $\mathfrak{o}_{/p}$ are both $\geq \mathfrak{o}$, and that $\mathfrak{o}_p = \mathfrak{o}_{/p}$ whenever either one is $\leq \mathfrak{o}+p-1$. In fact there are no further relations among these prevaluations. This can be seen using the examples $h^i + h^j p + h^k \varphi_p$ with $i \leq j, k$, which have $\mathfrak{o} = i$, $\mathfrak{o}_p = j + p - 1$ and $\mathfrak{o}_{/p} = k + p - 1$.

Proof of (3.4). First observe that f is a power of t times a polynomial in $\mathbb{Z}[t]$ with nonzero constant term. Since powers of t are units in Λ, which have value 0 with respect to any positive prevaluation on Λ, it suffices to prove the lemma for $f \in \mathbb{Z}[t]$ with $\mathfrak{o}(f) = 0$. In the language of ideals, it says (a) $\mathcal{O}^d \subset \mathcal{O}^d_p$ for all d, (b) $\mathcal{O}^d_p = \mathcal{O}^d_{/p}$ for $d < p$, and (c) $(\mathcal{O}^d_p)' \subset \mathcal{O}^{d-1}_p$ for $d < p$, where for brevity we denote the ideals $\mathcal{O}^d_{\mathfrak{o}_-}$ by $\mathcal{O}^d_-$.

Since $\mathbb{Z}[t]$ is identified with the polynomial ring $\mathbb{Z}[h]$ under the substitution $t = 1+h$, the lemma is really a statement about the prevaluations $\mathfrak{o}$, $\mathfrak{o}_p$ and $\mathfrak{o}_{/p}$ on $\mathbb{Z}[h]$, where $\mathcal{O}^d = (h^d)$, $\mathcal{O}^d_p = (h^d, \varphi_p)$ and $\mathcal{O}^d_{/p} = (h^d, p)$. Now (a) is obvious and (c) follows from (b) using the elementary observation that the derivative of any polynomial in $\mathcal{O}^d_{/p}$ is in $\mathcal{O}^{d-1}_{/p}$, since $d < p$. For (b) it suffices to show $\varphi_p \in \mathcal{O}^{p-1}_{/p}$ and $p \in \mathcal{O}^{p-1}_p$. But this follows immediately from the observation that φ_p and p are associates in the ring $\mathbb{Z}[h]/(h^{p-1})$, which can be seen as follows: By (13) $\varphi_p = pu + h^{p-1}$, where $u = \frac{1}{p}\sum_{k=1}^{p-1}\binom{p}{k}h^{k-1}$, and evidently $u = 1 + O(h)$ is a unit in $\mathbb{Z}[h]/(h^{p-1})$. $\square$

Bounds for the p-order

As a trivial consequence of Theorem 2.5 and Lemma 3.4a we obtain lower bounds for the p-orders of Ohtsuki polynomials of links in the 3-sphere.

Proposition 3.5. *Let L be an ℓ-component link of Milnor degree d. Then for any odd prime p,*

$$\mathfrak{o}_p(\phi_L) \geq 2\ell d/(d+1).$$

Furthermore, if L is diagonal with maximum cabling index m, then $\mathfrak{o}_p(\phi_L) \geq \ell + m$.

The following result of Ohtsuki [O1,§7.2] gives bounds on the p-orders of the p-sums $(a|c)$ defined in (4), for arbitrary integers a and c (see also [M1,§5.4]).

Proposition 3.6. (Ohtsuki) *Let $p = 2n+3$ be an odd prime. Then $\mathfrak{o}_p(a|c) \geq n-c$.*

This result can be strengthened when $a \equiv 0 \pmod p$ using Lemma 3.4.

Proposition 3.7. *Let $p = 2n+3$ be an odd prime and a be a multiple of p. Then $\mathfrak{o}_p(a|c) \geq 2(n-c)$.*

Proof. First observe that $(a|c)$ depends only on the residue class of $a \pmod p$, and so $(a|c) = (0|c)$. For convenience write (c) for $2(a|c)$ and set $m = n+1 = (p-1)/2$. Thus $(c) = \sum_{k=-m}^{m}[k][k,c)$, and it suffices to prove $\mathfrak{o}_p(c) \geq 2(n-c)$ since $\mathfrak{o}_p(2) = 0$. If $c < 0$, then $[k,c) = 0$ and so $\mathfrak{o}_p(c) = \infty$. If $c > n$, then $2(n-c) < 0$ and there is nothing to prove, since $\mathfrak{o}_p$ is positive. Thus we assume $0 \leq c \leq n$.

Now consider the more general sum $(j,c) = \sum_{k=-m}^{m}[k]^{(j)}[k,c)$ where ${}^{(j)}$ denotes the jth derivative (with respect to t or h). We will show

$$\mathfrak{o}_p(j,c) \geq 2(n-c) - j \tag{14}$$

(for $0 \leq c \leq n$) by double induction on c and j. The proposition follows by setting $j = 0$.

Initial step of the induction

It must be shown that $\mathfrak{o}_p(j,0) \geq 2n-j$, where $(j,0) = \sum_{k=-m}^{m}[k]^{(j)}[k]$. By Lemma 3.4b it suffices to prove

$$\mathfrak{o}_{/p}(j,0) \geq 2n-j. \tag{15}$$

To facilitate the proof, consider the ring $P = \mathbb{Q}[\cdot]$ of polynomials with rational coefficients in an un-named variable $\cdot$ and let P_{odd} be the subring of all odd polynomials. Set $\mathbb{P} = P[[h]]$ and $\mathbb{P}_{\text{odd}} = P_{\text{odd}}[[h]]$. For each integer $r \geq 0$, let P^r denote the subring of P consisting of all polynomials of degree $\leq r$ with coefficients in $\mathbb{Z}[1/r!]$, and $\mathbb{P}^r$ denote the subring of $\mathbb{P}$ consisting of power series whose coefficient of h^d is in P^{d+r} for each d. Set $P^r_{\text{odd}} = P^r \cap P_{\text{odd}}$ and $\mathbb{P}^r_{\text{odd}} = \mathbb{P}^r \cap \mathbb{P}_{\text{odd}}$.

Claim. $(j,0) = f(p)$ for some f in $\mathbb{P}^{j+3}_{\mathrm{odd}}$.

Here $f(p)$ denotes the power series obtained by plugging in p for the un-named variable in each coefficient of f. The desired inequality (15) is immediate from the claim and the following easy result.

Lemma 3.8. *If $f \in \mathbb{P}^r_{\mathrm{odd}}$ and $f(p)$ has* integer *coefficients, then $\mathfrak{o}_{/p} f(p) \geq p - r$.*

Proof of (3.8). Under the hypotheses, the coefficient of h^d in $f(p)$ is an integer of the form $f_d(p)$ for some *odd* polynomial f_d of degree $\leq d + r$ with coefficients in $\mathbb{Z}[1/(d+r)!]$. If $d < p - r$, then $d + r < p$ and so $f_d(p)$ is divisible by p. Hence $\mathfrak{o}_{/p}(f(p)) \geq p - r$. $\square$

To prove the claim, first observe that $[\cdot] \in \mathbb{P}^1$. In other words the coefficient $[k]_d$ of h^d in $[k]$ (viewed as a power series) is a polynomial in k of degree $\leq d+1$ with coefficients in $\mathbb{Z}[1/(d+1)!]$. This can be seen using calculus and the substitution $t = \exp(h/4)$ (see Remark 1.2) or by appealing to the following useful lemma. To state it, recall the "discrete derivative" $\Delta : P^{r+1} \to P^r$ defined by $\Delta F(\cdot) = F(\cdot + \frac{1}{2}) - F(\cdot - \frac{1}{2})$. Now for any polynomial $f \in P^r$, consider the function $\Sigma f : \mathbb{Z} \to \mathbb{Q}$ given by the sum

$$\Sigma f(k) = \sum_{j=-m}^{m} f(j) \tag{16}$$

where $m = (k-1)/2$. The point of the lemma is first to show that discrete integrals exist and are unique up to constants, and then to use this to show that $\Sigma f(\cdot)$ is in disguise a polynomial in P^{r+1}_{odd}.

Lemma 3.9. *Given $f \in P^r$, there exist unique $F, G \in P^{r+1}$ with G odd such that $\Delta F = f$, $F(0) = 0$, and $G(k) = \Sigma f(k)$ for every integer k.*

Proof of (3.9). For each integer $d \geq 0$ define $F_d \in P^d$ by $F_d(x) = x^d/d$, and set $f_d = \Delta F_{d+1}$. Clearly f_d is a monic polynomial of degree d with coefficients in $\mathbb{Z}[1/(d+1)!]$, and so $f = \sum_{d=0}^r a_d f_d$ for suitable $a_d \in \mathbb{Z}[1/(r+1)!]$. Thus $F = \sum_{d=0}^r a_d F_{d+1}$ is in P^{r+1} with $\Delta F = f$ and $F(0) = 0$. If E is any other polynomial with $\Delta E = f$, then $\Delta(E - F) = 0$. But then the polynomial $E - F$ is periodic, and therefore constant, whence $E = F$ if $E(0) = 0$.

The sum in (16) defining Σf telescopes when ΔF is substituted for f, and so equals $F(k/2) - F(-k/2)$. Thus the polynomial G defined by $G(x) = F(x/2) - F(-x/2)$ satisfies the last equality, and is unique since any polynomial is determined by its values on the integers. Clearly G is odd. $\square$

The fact that $[\cdot]$ is in $\mathbb{P}^1$ can now be seen as follows: Lemma 3.9 shows that Σ can be viewed as an operator $P^r \to P^{r+1}_{\text{odd}}$. This operator extends coefficient-wise to an operator $\Sigma : \mathbb{P}^r \to \mathbb{P}^{r+1}_{\text{odd}}$. Substitute $1+h$ for t in $[k] = (t^{2k} - t^{-2k})/(t^2 - t^{-2}) = \sum_{j=-m}^{m} t^{4j}$ and apply the binomial theorem to get

$$[k] = \sum_{d \geq 0} \sum_{j=-m}^{m} \binom{4j}{d} h^d$$

(the binomial coefficient is defined for *all* j by $4j(4j-1)\cdots(4j-d+1)/d!$ if $d > 0$, and 1 if $d = 0$). Evidently $\binom{4\cdot}{d} \in P^d$ and $[\cdot]_d = \Sigma\binom{4\cdot}{d}$. Thus $[\cdot]_d \in P^{d+1}_{\text{odd}}$, and so $[\cdot] \in \mathbb{P}^1$ by definition.

Now the product of series in $\mathbb{P}$ induces bilinear operators $\mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{r+s}$, and the jth derivative (with respect to h) induces operators $\mathbb{P}^r \to \mathbb{P}^{r+j}$. It follows that the series $\Sigma([\cdot]^{(j)}[\cdot])$ is in $\mathbb{P}^{j+3}_{\text{odd}}$. But $(j,0) = (\Sigma([\cdot]^{(j)}[\cdot])(p)$, by definition. This establishes the claim and thus completes the initial step of the induction.

Inductive step

Differentiating the cabled quantum integers gives $[k, c-1]' = c[2]'[k,c)$, which leads by a simple calculation to the formula $(j,c) = ((j,c-1)' - (j+1,c-1))\,/(c[2]')$. Now $[2] = t^2 + t^{-2}$, so $[2]' = 2t - 2t^{-3} = 8h + \cdots$ has p-order 1. Thus $\mathfrak{o}_p(c[2]') = 1$, since $\mathfrak{o}_p(c) = 0$ (note that $0 < c < m$ by assumption).

By the inductive assumption and Lemma 3.4c, $\mathfrak{o}_p(j, c-1)' \geq 2(n-c+1) - j - 1 = 2(n-c) - j + 1$ and $\mathfrak{o}_p(j+1, c-1) \geq 2(n-c+1) - (j+1) = 2(n-c) - j + 1$. Since $\mathfrak{o}_p$ is a prevaluation, it follows that $\mathfrak{o}_p(j,c) \geq 2(n-c) - j$, proving (14) and thus the proposition. $\square$

Combining Propositions 3.5–3.7, we obtain bounds on the orders of the p-brackets of links in the 3-sphere.

Theorem 3.10. *Let L be a framed link in the 3-sphere with framings $a_1, \cdots, a_\ell$. Then for any odd prime $p = 2n+3$,*

$$\mathfrak{o}_p\langle L\rangle \geq (\ell + \frac{d-1}{d+1}\, b)\, n$$

where d is the Milnor degree of L and b is the number of framings divisible by p. For L diagonal ($d \geq 2$) and $b > 0$, the bound $\mathfrak{o}_p\langle L\rangle \geq (\ell+1)n$ holds as well.

Proof. Order the components of L so that p divides a_i for $i \le b$. For each coloring $c = (c_1, \ldots, c_\ell)$ of L set $c_{\max} = \max(c_i)$, $|c| = \sum_{i=1}^{\ell} c_i$ and $|c|_p = \sum_{i=1}^{b} c_i$. Since $\mathfrak{o}_p$ is a prevaluation, it follows from Proposition 1.5 that $\mathfrak{o}_p\langle L\rangle$ is bounded below by the minimum over all colorings $0 \le c \le n$ of $\sum_{i=1}^{\ell} \mathfrak{o}_p(a_i|c_i) + \mathfrak{o}_p(\phi_{L^c})$. Applying the bounds for $\mathfrak{o}_p(a_i|c_i)$ in Propositions 3.6–3.7 and the bound for $\mathfrak{o}_p(\phi_{L^c})$ in Proposition 3.5 gives

$$\mathfrak{o}_p\langle L\rangle \ge (\ell + b)n + \min_{0\le c\le n} \left(\frac{d-1}{d+1}|c| - |c|_p\right).$$

The minimum is clearly achieved when the first b components have the maximum allowable cabling index n, and the remaining components have index 0, whence $|c| = |c|_p = bn$. The first inequality in the theorem is now immediate.

If L is diagonal, then $\mathfrak{o}_p\langle L\rangle \ge (\ell + b)n + \min(c_{\max} - |c|_p)$ by the alternative bound given in Proposition 3.5. For $b > 0$, this expression is minimized for the same cabling as before, and so $c_{\max} = n$ and $|c|_p = bn$. Therefore in this case $\mathfrak{o}_p\langle L\rangle \ge (\ell + 1)n$. □

Exact values of the p-order

Computational evidence suggests that the bounds for the p-orders of the p-sums $(a|c)$ given in Propositions 3.6 and 3.7 may be sharp:

Question Q(a|c). *Let $p = 2n + 3$ be an odd prime, a and c be integers with $0 \le c \le n$. Set $r = 2$ if p divides a, and $r = 1$ otherwise. Is $\mathfrak{o}_p(a|c) = r(n - c)$?*

Ohtsuki's work [O1,§3.6], together with the improvements in [LW], suggest a positive answer to $Q(\pm 1|c)$; one must show that the invariants $\nu_{\pm 1,c,0}$ of [LW] are not divisible by p, as verified there for $c \le 2$.

The answer to $Q(a|0)$ is yes. To see this, recall that the p-sum $(a|0)$, or equivalently the p-bracket b_a of the a-framed unknot, is given by

$$b_a = \sum_{k=1}^{p/2} t^{a(k^2-1)}[k]^2.$$

Closed forms for these sums as elements of Λ_p are well known, and can be obtained by elementary computations involving Gauss sums. (Up to a factor, b_a coincides with the quantum invariant of the lens space $L(a, 1)$; formulas for general lens spaces in terms of Dedekind sums are known as well [KM2] [Je] [Ga].) In particular it will be seen below that b_a is an associate of h^{rn} in Λ_p, and so $\mathfrak{o}_p(b_a) = rn$. The exact form of b_a will not concern us here, except when $a \equiv 0 \pmod p$.

Proposition 3.11. *Let $p = 2n + 3$ be an odd prime and a be an arbitrary integer. Then $\mathfrak{o}_p(b_a) = rn$, where $r = 1$ or 2 according to whether a is prime to p or not. In fact there exists a unit u_a in Λ_p, depending only on the mod p residue class of a, such that $b_a = u_a h^{rn}$. Furthermore $(-1)^n u_\mathrm{o}$ is a square in Λ_p, that is $u_\mathrm{o} = (-1)^n v_\mathrm{o}^2$ for some unit v_o in Λ_p.*

Proof. Clearly $[p-k] = -[k]$ and $t^{a((p-k)^2-1)} = t^{a(k^2-1)}$, and so the sum $\sum_{k=1}^{p/2}$ in the definition of b_a is half of the complete sum $\sum_{k=1}^{p}$, denoted simply by $\sum$ below.

First consider the case when a is prime to p, and calculate b_a "up to units" (using the notation $x \sim y$ to indicate that x and y are associates). Since $h \sim z = s - s^{-1}$, we have $2h^2 b_a \sim \sum t^{ak^2}(s^k - s^{-k})^2 = \sum(t^{ak^2+4k} - 2t^{ak^2} + t^{ak^2-4k})$. Completing the square gives $\sum(t^{a(k+2\bar{a})^2-4\bar{a}} - 2t^{ak^2} + t^{a(k-2\bar{a})^2-4\bar{a}})$, where $\bar{a}$ is the mod p inverse of a. Since the sum is over a complete set of residues mod p, the quadratic part of each of the three terms in this expression contributes a Gauss sum $G_a = \sum t^{ak^2}$. Factoring this out gives $2G_a(t^{-4\bar{a}}-1) \sim 2Gh$, since $h \sim t^k - 1$ for any k prime to p and $G = \pm G_a \sim h^{n+1}$ (see Remark 3.1a). Thus $b_a \sim G/h \sim h^n$.

The exact calculation of b_o is easy, involving only the sum of a geometric series. We have $2z^2 b_\mathrm{o} = \sum(s^k - s^{-k})^2 = \sum(q^k - 2 + q^{-k}) = -2p$, since the sum of all the pth roots of unity vanishes. By Remark 3.1a, $p = (-1)^m [m]!^2 z^{2m}$, and so $b_\mathrm{o} = -p/z^2 = (-1)^n [m]!^2\, z^{2n} = ((-1)^n [m]!^2 (z/h)^{2n})\, h^{2n}$. Therefore $u_\mathrm{o} = (-1)^n v_\mathrm{o}^2$ for $v_\mathrm{o} = [m]!(z/h)^n$. □

Remarks 3.12. (a) The computation $b_\mathrm{o} = -p/z^2$ in the preceding proof can be carried out just as easily for the family of sums

$$s_j = \sum_{k=1}^{p/2} [jk][k],$$

for which $s_1 = b_\mathrm{o}$. Indeed $2z^2 s_j = \sum(s^{(j+1)k} - s^{(j-1)k} - s^{-(j-1)k} + s^{-(j+1)k})$. If $j \not\equiv \pm 1 \pmod{p}$ then $s^{j\pm1}$ are both primitive pth roots of unity, and so the sum vanishes. For $j \equiv \pm 1$, only two of the four terms survive, which sum to $\mp 2p$. Thus $s_j = \pm b_\mathrm{o}$ for $j \equiv \pm 1 \pmod{p}$, and $s_j = 0$ otherwise. These sums will appear in the examples in §5.

(b) Another family of sums which arise in §5 and are easily computed is defined by

$$t_a = \sum_{k=1}^{p/2} s^{a(k^2-1)}[k^2].$$

(Again these depend only on the mod p residue of a.) Proceeding as above, we have $2zs^a t_a = \sum(s^{(a+1)k^2} - s^{(a-1)k^2}) = G_{2(a+1)} - G_{2(a-1)}$, where G_0 is the "degenerate" Gauss sum $\sum 1^{k^2} = p$. This difference is easily analyzed according to the quadratic nature of $a \pm 1 \pmod{p}$, and yields either $\pm G \pm p$ (when $a \equiv \pm 1$), 0 (when $a \pm 1$ are either both quadratic residues or both non-quadratic residues), or $\pm 2G$ (otherwise). In particular

$$t_{\pm 1} = \frac{(\pm 1)^m G - p}{2(q^{\pm 1} - 1)}$$

where $m = (p-1)/2$. This has the same p-order as $G/h \sim h^n$, and so $\mathfrak{o}_p(t_{\pm 1}) = n$. Note that $t_1 \neq t_{-1}$, which has an interesting topological application (see §5.5).

(c) The p-orders of the trivial sums $u = \sum_{k=1}^{p/2} 1$ and $v = \sum_{j \le k=1}^{p/2} 1$ are both zero. Indeed $u = m$ and $v = m(m+1)/2$ are both relatively prime to p. More generally for any function $f(k)$ or $g(j,k)$, the sums $u_f = \sum_k t^{f(k)} = u + O(h)$ and $v_g = \sum_{j \le k} t^{g(j,k)} = v + O(h)$ have zero p-order. Note that if $c = \sum_k f(k)$ is also prime to p, then $u_f \neq u_{-f}$, since the linear coefficient in $u_{\pm f} = \pm c$. A similar statement holds for v_g, which has an interesting topological application (see §5.5).

§4. Quantum invariants

Fix an odd prime $p = 2n+3$ and a closed oriented 3-manifold M. In this section we discuss the quantum $SO(3)$ invariant $\tau_p(M)$, which in the normalization given here is an element of the cyclotomic field Q_p, and give a lower bound for its order in terms of the mod p first betti number of M. We also introduce a collection of "finite type invariants" τ_p^d that dominate τ_p. These are studied in more detail in [CM1].

The 3-manifold invariant

Let L be an ℓ-component integrally framed link in the 3-sphere S^3. Write ℓ_0, ℓ_+ and ℓ_- for the number of zero, positive and negative eigenvalues, respectively, of the linking matrix A_L of L (with framings on the diagonal). Alternatively ℓ_0 can be viewed as the nullity of the integral quadratic form given by A_L. Define the *p-norm* of L to be the element

$$|L| = b_1^{\ell_+} b_{-1}^{\ell_-} (b_0/h^n)^{\ell_0} \tag{17}$$

in Q_p, where b_a is the p-bracket $\langle \bigcirc_a \rangle$ of the a-framed unknot and $h = t-1$ as usual (see (10) for the definition of p-bracket). By Proposition 3.11, $|L|$ lies in the ring of integers Λ_p in Q_p and is of the form

$$|L| = u h^{n\ell} \tag{18}$$

for some unit $u \in \Lambda_p$ (namely $u_1^{\ell_+} u_{-1}^{\ell_-} u_0^{\ell_0}$). In particular $\mathfrak{o}_p|L| = n\ell$.

Now consider the 3-manifold S^3_L obtained by surgery on L. Since every 3-manifold arises in this way for some link, we may assume $M = S^3_L$. Note that the nullity ℓ_0 is an invariant of M, namely the first Betti number $b(M) = \mathrm{rk}(H_1(M))$, since A_L is a presentation matrix for $H_1(M)$. Similarly the nullity of A_L as a form over $\mathbb{Z}_p$ is the mod p first Betti number $b_p(M) = \mathrm{rk}(H_1(M;\mathbb{Z}_p))$. If L is diagonal, then $b(M)$ is the number of components of L with framing 0, and $b_p(M)$ is the number with framings divisible by p.

Define the *level p quantum $SO(3)$ invariant* of M to be the quotient

$$\tau_p(M) = \langle L\rangle/|L| \tag{19}$$

where $\langle L\rangle$ is the p-bracket viewed as an element of Λ_p. (Here p need not be prime, although it must be odd to give a well defined invariant.) Clearly $\tau_p(M)$ lies in $\Lambda_p[h^{-1}] \subset Q_p$, by (18), and in fact in Λ_p as will be seen below. It is also clear that the invariant τ_p is *multiplicative* under connected sums, $\tau_p(M\#N) = \tau_p(M)\tau_p(N)$, since the bracket is multiplicative and $\ell_0, \ell_\pm$ are additive under distant unions of links (cf. [KM1,§5.9]). The motivation for calling it the "$SO(3)$" invariant arises in the quantum group setting: the p-bracket of L can be rewritten using the "Symmetry Principle" ([KM1,§4.20]) as the sum over *odd* colorings $k < p$, which correspond to those representations of $SU(2)$ which factor through $SO(3)$ (cf. [BHMV]).

Remark 4.1. This invariant was first defined in [KM1,§8.10] with a slightly different normalization, denoted there by τ_p'. In particular $\tau_p'(M) = \langle L\rangle/|L|'$ (evaluated at $t = \exp(2\pi i m^2/p)$) where $|L|' = b_{+1}^{\ell_+} b_{-1}^{\ell_-} b_0^{\ell_0/2}$. This definition is shown independent of the choice of L by establishing the invariance of $\langle L\rangle$ under "handle-slides", one of the two moves in the Kirby calculus relating any two framed links which give the same 3-manifold [Ki]. (See [L2] for a purely skein theoretic proof of this invariance.) It is then an elementary exercise to show that invariance under "blow-ups" (the other move) is achieved by dividing by *any* factor of the form $|L|_c = b_{+1}^{\ell_+} b_{-1}^{\ell_-} c^{\ell_0}$ (c can be an arbitrary constant since ℓ_0 is an invariant of M). In particular the constants for τ_p and τ_p' are $c = b_\mathrm{o}/h^n$ and $c' = b_\mathrm{o}^{1/2}$.

Of course the choice of constant affects what properties the quantum invariants have. For example τ_p' is involutive, i.e. conjugates under orientation reversal, whereas τ_p is not. On the other hand (and this is what motivated our choice of normalization) τ_p takes its values in Q_p, whereas τ_p' does not in general because of the square root in c'. However τ_p' does take values in $Q_{4p} = Q_p(\sqrt{-1})$ which differ by units from the corresponding values of τ_p (assuming p is prime). Indeed Proposition 3.11 shows that $c = (-1)^n v_\mathrm{o}^2 h^n$ and $c' = \varepsilon v_\mathrm{o} h^n$, where $\varepsilon = (-1)^{n/2}$, so $\tau_p(M) = (\epsilon v_\mathrm{o})^{b(M)}\tau_p'(M)$.

Quantum cyclotomic orders

Define the *quantum p-order* $\mathfrak{o}_p(M)$ of M to be p-order of $\tau_p(M)$, that is

$$\mathfrak{o}_p(M) = \mathfrak{o}_p(\tau_p(M)) \tag{20}$$

where the $\mathfrak{o}_p$ on the right is the p-adic valuation on Q_p defined in §3. Observe that $\mathfrak{o}_p$ is *additive* under connected sums, since τ_p is multiplicative.

There is a simple relation between the quantum p-order of $M = S^3_L$ and the p-order of the p-bracket $\langle L\rangle$:

$$\mathfrak{o}_p(M) = \mathfrak{o}_p\langle L\rangle - n\ell. \tag{21}$$

Indeed $\mathfrak{o}_p(M) = \mathfrak{o}_p\langle L\rangle - \mathfrak{o}_p|L|$, since $\mathfrak{o}_p$ is a valuation, and $\mathfrak{o}_p|L| = n\ell$ by (18). Now the general bound $\mathfrak{o}_p\langle L\rangle \geq n\ell$ of Theorem 3.10 implies that $\mathfrak{o}_p(M) \geq 0$. Since $\tau_p(M) \in \Lambda_p[h^{-1}]$, this recovers Murakami's integrality result:

Theorem 4.2. ([M1] [MR]) *τ_p takes values in the cyclotomic ring Λ_p.*

The stronger bounds in Theorem 3.10 lead to corresponding bounds for the quantum p-orders of 3-manifolds, and then combining these with a result of Murakami and Ohtsuki, to the universal betti number bound alluded to in the introduction. Define the *Milnor degree* of a 3-manifold M to be the *maximum* Milnor degree of all the (integrally) framed links L for which $M = S^3_L$, and call M *diagonal* if it has Milnor degree ≥ 2.

Theorem 4.3. *Let M be a closed oriented 3-manifold and $p = 2n+3$ be an odd prime. Then $\mathfrak{o}_p(M) \geq b_p(M)\, n(d-1)/(d+1)$, where $b_p(M) = \mathrm{rk}(H_1(M;\mathbb{Z}_p))$ and d is the Milnor degree of M. In particular for diagonal M (meaning $d \geq 2$),*

$$\mathfrak{o}_p(M) \geq b_p(M)\, n/3.$$

In fact this bound holds for any *M. Furthermore $\mathfrak{o}_p(M) \geq n$ if $b_p(M)$ is positive.*

Proof. The first inequality is immediate from (21) and Theorem 3.10, as are the other two for the case when M is *diagonal*. But by the diagonalizing lemma of Murakami and Ohtsuki [M2,§2.3], there exists a diagonal 3-manifold N with $\mathfrak{o}_p(N) = b_p(N) = 0$ such that $M\#N$ is diagonal; in particular N may be taken to be a connected sum of lens spaces $L(k_i, 1)$ with k_i prime to p. Thus M has the same quantum p-order and mod p first betti number as a diagonal manifold (since both $\mathfrak{o}_p$ and b_p are additive under connected sums) and so both inequalities hold in general. □

The finite type invariants τ_p^d and further bounds for the quantum p-order

Let $\mathcal{M}$ denote the free abelian group generated by closed oriented 3-manifolds (up to oriented diffeomorphism). Theorem 4.2 shows that τ_p can be viewed as a $\mathbb{Z}$-linear map

$$\tau_p : \mathcal{M} \to \Lambda_p.$$

Due to the well known computational complexity of τ_p, one would not expect this map to be of "finite type" in any natural sense. It turns out, however, that τ_p is a *limit* of finite type invariants, in the sense of the theory developed by the authors in [CM1].

Recall from [CM1] that a $\mathbb{Z}$-linear map from $\mathcal{M}$ to an abelian group A is said to be of *finite type* if there exists an integer $d \geq 0$ such that $\lambda(M_{\delta L}) = 0$ for all *admissible* pairs (M,L) with $\ell > d$; the smallest such d is called the *degree* of λ. Here L is an ℓ-component framed link in a 3-manifold M, and admissibility means that each component of L is null-homologous in M with framing ± 1 and zero linking number with any other component. The notation $M_{\delta L}$ represents the alternating sum of manifolds obtained by surgery on all the sublinks of L,

$$M_{\delta L} = \sum_{S<L} (-1)^s M_S$$

(cf. Remark 1.4).

Now the fact that τ_p is not of finite type (for $p \neq 3$ since $\tau_3 \equiv 1$) is an easy consequence of Murakami's beautiful formula $\tau_p(M) = 1 + 6\lambda(M)h + O(h^2)$ for homology spheres, where λ is the Casson invariant [M1]. Indeed it follows from this formula that $\tau_p(S^3_{\delta T_d}) \neq 0$ for any $d > 0$, where T_d is the distant union of d copies of the $+1$-framed right-handed trefoil.

It is shown in [CM1], however, that the maps

$$\tau_p^d : \mathcal{M} \to \mathbb{Z}_{p^k},$$

obtained from τ_p by composing with the projections $\pi^d : \Lambda_p \to \mathbb{Z}_{p^k}$ defined in (3.2), are of finite type (of degree $\leq 3d$, in fact $\leq 3d - b(p-3)/2$ when restricted to manifolds with $b_p = b$). It follows that τ_p is "dominated" by finite type invariants. In other words the value of $\tau_p(M)$ can be recovered from the values $\tau_p^d(M)$ for all d (just as one can recover an integer from its residues mod p^k for all k). For $\mathbb{Z}_p$-homology spheres M, this can also be deduced from the recent result of Rozansky [R2] (conjectured by Lawrence [La]) that $\tau_p(M)$ is the p-adic limit of the Ohtsuki series $\sum \lambda_n h^n$ [O1] [O2].

Remark 4.4. The degree zero invariants are familiar algebraic topological invariants. Indeed the equivalence relation on 3-manifolds generated by surgery on admissible links coincides with notion of H_1*-bordism* [Ge] [CGO,§2.8,3.1]. (Recall that 3-manifolds M and N are H_1-bordant if there is a 4-manifold W with boundary $M \cup -N$ such that the inclusions $M \to W \leftarrow N$ induce isomorphisms on H_1.) Thus the degree zero invariants are exactly the invariants of H_1-bordism, including for example the first betti number and the mod p first betti numbers for each p. In fact the H_1-bordism class of a 3-manifold has a characterization in terms of its cohomology ring and linking form (see [CGO,§3.1]). For example, the H_1-bordism class of a connected sum of b copies of $S^1 \times S^2$ consists of all 3-manifolds with $H_1 = \mathbb{Z}^b$ and with vanishing triple cup product form on H^1 [CGO,§3.6].

The perspective on the study of 3-manifolds suggested by the theory of finite type invariants leads to sharper bounds on the quantum p-order. As an illustration of this, we prove a result about the orders of manifolds H_1-bordant to a connected sum of $S^1 \times S^2$'s.

Proposition 4.5. *Let M be H_1-bordant to $\#^b(S^1 \times S^2)$. Then for any odd prime $p = 2n+3$, the quantum p-order $\mathfrak{o}_p(M) \geq bn/2$.*

Note that $\#^b(S^1 \times S^2)$ has order bn (as will be seen in §5) while Theorem 4.3 gives a lower bound of $bn/3$ for the order of any manifold with the same first betti number. Further more, the examples in the next section realize the lower bound of $\lceil b/3 \rceil n$ (which is generally much closer to $bn/3$ than to $bn/2$) for such manifolds. Thus the bound in the Proposition reflects a strong restriction on the orders of manifolds H_1-bordant (and thus "closer") to $\#^b(S^1 \times S^2)$, providing further insight on the topological nature of $\mathfrak{o}_p(M)$.

Proof of (4.5). By the remarks above, $M = S^3_L$ with $L = J \cup K$, where $J = \bigcirc^b$ (the zero-framed unlink of b components) and K is admissible in $S^3_J = \#^b(S^1 \times S^2)$ (i.e. K has null homologous components with zero linking numbers and ± 1 framings). It follows that K is admissible in S^3 and that each component of K has zero linking number with each component of J. Now using (21), and proceeding as in the proof of Theorem 3.10, we have

$$\mathfrak{o}_p(M) \geq bn + \min_{0 \leq c \leq n} (\mathfrak{o}(\phi_{L^c}) - |c| - |c|_p).$$

As in the proof of Theorem 2.5, the order of ϕ_{L^c} is bounded below by the minimum possible degree of a chordless Feynman diagram on L^c which has (1) at least two external vertices on each component of L^c, and (2) no trees with *all* their external vertices on the cabling of J (since J has infinite Milnor degree). In fact one need only consider diagrams with *exactly* two vertices on each component of L^c, since excess vertices can be traded for loops without affecting the degree (see Figure 1). Such a diagram will be called *admissible.*

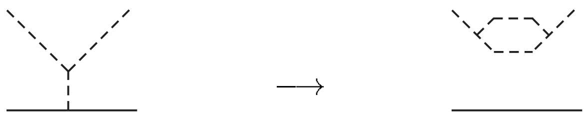


Figure 1

Set $j = j(c) = 2|c|_p$ and $k = k(c) = 2(|c| - |c|_p)$, representing the number of external vertices in admissible diagrams on L^c which lie on the cabling of J and K, respectively. (Note that $j \leq 2bn$ since $c \leq n$.) Then

$$\mathfrak{o}_p(\phi_{L^c}) - |c| - |c|_p \geq k/2 - t(c)$$

where $t(c)$ is the maximum possible number of trees in an admissible diagram on L^c (cf. the proof of Theorem 2.5). Since chords are disallowed, $t(c) \leq y(c) + (j+k-3y(c))/4 = (j+k+y(c))/4$ where $y(c)$ is the maximum possible number of y's (dashed degree 2 trees) in an admissible diagram on L^c. Because of condition (2) above, every y must have atleast one vertex on the cabling of K, and so it is clear that $y(c)$ is bounded above by the function $f(j,k) = \min(k,(j+k)/3)$. (Note that f assumes the first value of the minimum if $j \geq 2k$, and the second value if $j \leq 2k$.) Thus $t(c) \leq (j+k+f(j,k))/4$.

It follows that

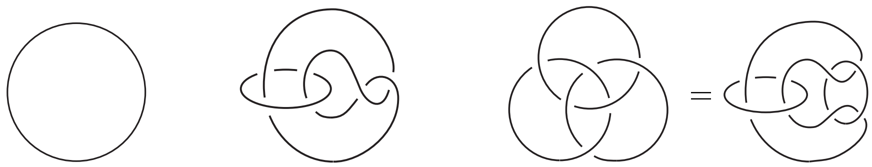
$$\mathfrak{o}_p(M) \geq bn + \min_{0 \leq c \leq n} g(j,k)$$

where $g(j,k) = (k - j - f(j,k))/4$. One readily computes $g(j,k) = (k-2j)/6$ for $j \leq 2k$, and $g(j,k) = -j/4$ for $2k \leq j \leq 2bn$. Since $\partial g/\partial j$ is negative, and $\partial g/\partial k$ is positive for $j < 2k$ and zero for $j > 2k$, g assumes its minimum value of $-bn/2$ when $j = 2bn$ (the maximum allowable value of j) and $k \leq bn$. Therefore $\mathfrak{o}_p(M) \geq bn/2$. $\square$

Remark 4.6. The previous proposition can also be deduced from Theorem 4.3, since it is known that a 3-manifold M with $H_1(M)$ torsion free is H_1-bordant to a connected sum of $S^1 \times S^2$'s if and only the Milnor degree $d(M) \geq 3$ [CGO, §§3.5 and 6.10]. In a future paper, we will elaborate on this point of view, giving a characterization $d(M)$ in terms of Massey products. In particular, it will be shown that $d(M) = w - 1$ where w is the weight of the first nonvanishing Massey product of M [CM2].

§5. Examples

In this section it is shown that the general betti number bounds established in Theorem 4.3 are sharp. To accomplish this, we use examples constructed from three familiar framed links: the unknot L_1, the (left-handed) Whitehead link L_2, and the Borromean rings L_3 (see Figure 2), all equipped with the zero-framing.



(a) L_1 (b) L_2 (c) L_3

Figure 2

Observe that L_2 and L_3 have Milnor degree 3 and 2, respectively, while L_1 has infinite Milnor degree. It follows from Theorem 4.3 that the quantum p-orders of the 3-manifolds $M_\ell = S^3_{L_\ell}$, obtained by surgery on these framed links, are all $\geq n$. In fact these manifolds are all of p-order *exactly* n, as will be seen below. (This result is well known for $M_1 = S^1 \times S^2$ and $M_3 = T^3$.)

Theorem 5.1. *For any odd prime $p = 2n+3$, the 3-manifolds M_1, M_2 and M_3 all have quantum p-order n.*

Since $b_p(M_\ell) = \ell$, it follows that the bound $\mathfrak{o}_p \geq n$ is sharp for $b_p = 1$, 2 or 3. Furthermore $\mathfrak{o}_p(\#^k M_3) = kn$, since $\mathfrak{o}_p$ and b_p add under connected sums. Therefore the bound $\mathfrak{o}_p \geq b_p n/3$ is sharp for $b_p \equiv 0 \pmod 3$.

Remark 5.2. It is not known whether the betti number bound $\mathfrak{o}_p \geq b_p n/3$ is sharp for all b_p. To show this, it would suffice to produce two 3-manifolds M_ℓ (for $\ell = 4, 5$) with $b_p(M_\ell) = \ell$ and $\mathfrak{o}_p(M_\ell) = \lceil \ell n/3 \rceil$, where $\lceil\ \rceil$ is the least integer upper bound function; possible candidates are surgeries on zero framed links L_ℓ with $\bar{\mu}_{ijk} = 1$ for all distinct triples i, j, k (i.e. every three component sublink has a "Borromean interaction"). The sharpest betti number bound that can be deduced from the examples in this paper is $o_p \geq \lceil b/3 \rceil n$, realized by connected sums of copies of M_2 (at most two) and M_3.

Proof of (5.1). By (21) it suffices to show $\mathfrak{o}_p \langle L_\ell \rangle = (\ell + 1)n$ for $\ell = 1$, 2 and 3. This is immediate for $\ell = 1$ since $\langle L_1 \rangle = b_{\mathrm{o}}$, which has p-order $2n$ by Proposition 3.11. For $\ell = 2$ and 3, this will be accomplished by expressing $\langle L_\ell \rangle \in \Lambda_p$ in terms of b_{o} and the sums discussed in Remark 3.12. In particular, it will be shown that $\langle L_2 \rangle = b_{\mathrm{o}} t_1$ and $\langle L_3 \rangle = b_{\mathrm{o}}^2 u$, which will prove the theorem.

Recall from (10) that the p-bracket $\langle L \rangle$ of any zero-framed link L is given by the linear combination $\sum_{k=1}^{p/2} [k] J_{L,k}$ of colored Jones polynomials $J_{L,k}$. By allowing link colorings in the group ring $\Lambda_p \mathbb{Z}$ and expanding multilinearly, $\langle L \rangle$ can be viewed as a single colored Jones polynomial $J_{L,\omega}$ with each component colored by

$$\omega = \sum_{k=1}^{p/2} [k] k \in \Lambda_p \mathbb{Z}.$$

This point of view was introduced by Lickorish [L1] and will be adopted here. In particular a *coloring* will henceforth mean a $\Lambda_p\mathbb{Z}$-coloring (which of course need not assign the same color to each component).

Now consider the equivalence relation $\approx$ on the set of Λ_p-linear combinations of colored links, defined by $(L, \lambda) \approx (L', \lambda')$ iff $J_{L,\lambda} = J_{L',\lambda'}$. For example the following local equivalences involving integer colors (indicated by labels j and k) are well known:

(22) 

cf. [KM1,§3.27]. For simplicity it is assumed that $0 < j, k < p/2$.

It follows that

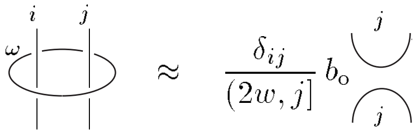
(23) 

where δ is the Kronecker delta and b_{o} is the p-bracket of the zero-framed unknot. Indeed the (23a) is immediate from (22a) and the definition of ω. The equivalence

(23b) says that the colored Jones polynomial of the left-hand link vanishes unless $j = 1$, in which case it equals b_o times the polynomial of the link obtained by removing the two pictured components. This follows from Remark 3.12a and the obvious fact that 1-colored components can be ignored, since the left hand side reduces to $[j]^{-1} s_j \,|^{j}$ using (22b) (also see [L2, Lemma 6]).

The following generalization of (23) provides the key to calculating the p-bracket $\langle L_\ell \rangle = J_{L_\ell, \omega}$:

(24)



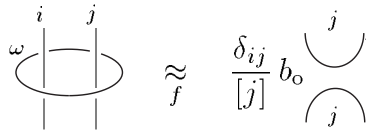
Here $(2w, j]$ is the framed quantum integer $s^{w(j^2-1)}[j]$, where w is defined by the following scheme. Observe that in the left-hand link, the two vertical strands are either a) oppositely oriented arcs on the same component, b) identically oriented arcs on the same component, or c) arcs from distinct components. For each of these cases in turn, define w to be

a) $lk(\ \ )$ b) $2\,lk(\ \ ) - 1$ c) $lk(\ \ )$

where the picture specifies a two component oriented link formed by locally modifying the component(s) containing the vertical strands.

The framed version of (24) is well known and has appeared in various forms in the literature (see for example [Tu,§II.3.10]). In particular, one defines the Jones polynomial of a *framed* colored link (L, k) to be $t^{a(k^2-1)} J_{L,k}$, where a is the framing, with associated equivalence relation $\underset{f}{\approx}$. Then (24) takes the diagramatic form

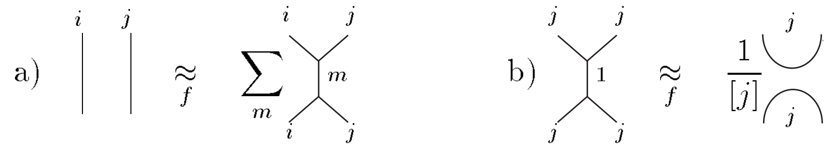
(25)



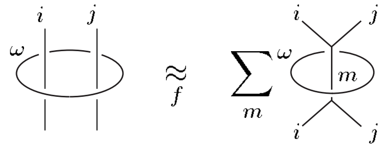
with respect to the "blackboard framing", since the sum of the blackboard framings of the components on the left-hand side exceeds the corresponding sum on the right-hand side by exactly $2w$.

The proof of (25) is easiest in the general context of framed colored *tri-valent graphs* (cf. [KR] [KL]). In the quantum group approach, the Jones polynomial of such a graph is defined locally in terms of operators assigned to the elementary tangles |, ∪, ∩, ╳, ╳, Y and ⅄. Specifically Y represents a natural injection of a simple $U_q(sl_2)$-module into the tensor product of two other simple modules, and ⅄ represents the corresponding projection. These satisfy the following identities:

(26)

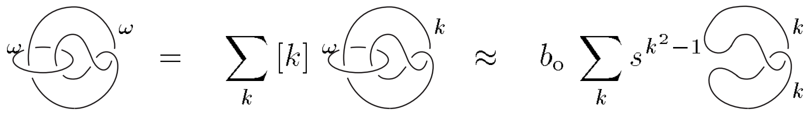


where the sum is over all *admissible* m, i.e. $|i-j| < m < i+j$ with $i+j+m$ odd (see [KR,§4.11-12]). By (26a)

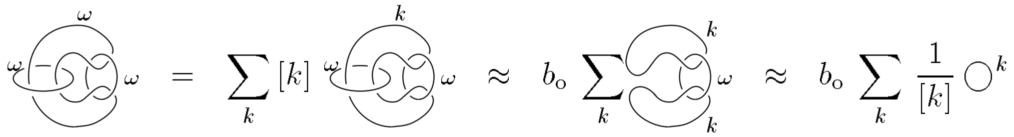


which vanishes by (23b) unless $m = 1$. But 1 is admissible if and only if $i = j$, and so (25) follows from (23b) and (26b).

It is now straightforward to compute the p-brackets of L_2 and L_3. Applying (24) to (L_2, ω) gives



But $J_{k\,k} = [k^2]$ by (22), and so $\langle L_2 \rangle = b_\circ t_1$. Similarly, applying (24) twice to (L_3, ω) gives



and so $\langle L_3 \rangle = b_\circ^2 u$ since $J_{\bigcirc^k} = [k]$. $\square$

Given a 3-manifold M, let $r_p(M)$ denote the *minimum* quantum p-order of any manifold in its *betti class* (all manifolds with the same first betti number as M). The previous theorem gives a partial determination of $r_p(M)$, namely $r_p(M) = b(M)n/3$ if $b(M)$ is divisible by 3, and $r_p(M) = n$ if $b(M) \le 3$, where as usual $n = (p-3)/2$.

Remark 5.3. There is no *maximum* quantum p-order for manifolds in a given betti class. Indeed for any prime p, there exist manifolds N_b with first betti number b and infinite p-order (i.e. $\tau_p(N_b) = 0$). For example, if $p \equiv 1 \pmod 3$ then let N_1 be 0-surgery on the (right-handed) trefoil, or equivalently $(1,0)$-surgery on the (right-handed) Whitehead link. The p-bracket of this framed link can be calculated, as for $\langle L_2 \rangle$ in the proof above, to be $b_\circ t_{(p-1)/2}$, which has infinite p-order by Remark 3.12b. (Note that $(p-1)(p+3)$ is a square mod p under the congruence assumption on p, by an elementary calculation using quadratic reciprocity.) Now let $N_b = \#^b N_1$ for $b > 0$, and $N_0 = p$-surgery on the trefoil.

It is also an interesting problem to determine the minimum quantum p-order of any manifold in the H_1-*bordism class* (see Remark 4.4) of M, denoted $s_p(M)$. For example consider the family of links L_3^k obtained from the Borromean rings by replacing one of the components with its $(k,1)$-cable (L_3^{-2} is pictured in Figure 3a) and let M_3^k denote the 3-manifold obtained by zero-framed surgery on L_3^k. It is known that M_3^j and M_3^k are H_1-bordant if and only if $j = \pm k$, and that these manifolds represent all the H_1-bordism classes of manifolds with first homology $\mathbb{Z}^3$ ([CGO,§3.3]). By Theorem 4.3 and Proposition 4.5, $s_p(M_3^k) \geq n$ for all $k \neq 0$ and $s_p(M_3^0) \geq 3n/2$. We suspect that all of these bounds are sharp.

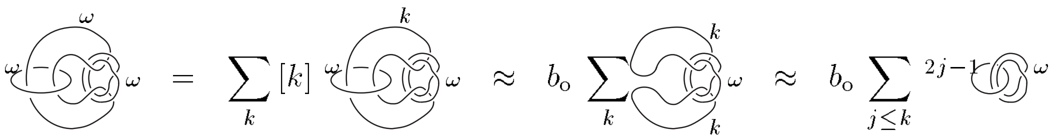
(a) (b)

Figure 3

Of course M_3^1 is just the 3-torus, and so $s_p(M_3^1) = n$ by the previous theorem. In contrast, $M_3^0 = \#^3 S^1 \times S^2$ has order $3n$, which is not minimal in its H_1-bordism class. Indeed the manifold $S^3_{L_1 \sqcup L_2} = M_1 \# M_2$ has order $2n$. Presumably the 3-manifold obtained by zero surgery on the link shown in Figure 3b has p-order $\lceil 3n/2 \rceil$, although this has not been verified. This would show that $s_p(M_3^0) = \lceil 3n/2 \rceil$. For $|k| > 1$, we suspect that M_3^k has p-order n, which would give $s_p(M_3^k) = n$ for $k \neq 0$. This can be verified for $k = \pm 2$ as follows.

Proposition 5.4. *For any odd prime $p = 2n+3$, the manifolds M_3^k have quantum p-order n for $k = \pm 2$.*

Proof. By (21) it suffices to show $\langle L_3^k \rangle = 4n$. Proceeding as in the proof of Theorem 5.1, this bracket (for $k = -2$) is equal to the colored Jones polynomial of



where the final sum is over $1 \leq j \leq k < p/2$. The last equivalence follows from a well known cabling principle for colored Jones polynomials (see for example [KM1,§3.10]) and the decomposition of tensor products of simple U_q-modules ([KM1,§2.13]). Since the final pictured link is symmetric, the equivalence (24) can be applied once more to give $\langle L_3^{\pm 2} \rangle = b_o^2 \sum_{j \leq k} q^{\mp 2j(j-1)}$, which has p-order $4n$ by Remark 3.12c. $\square$

Remark 5.5. The two 3-manifolds M_3^2 and M_3^{-2} (which are orientation reversing diffeomorphic) not only have the same p-orders, but also the same lowest order coefficients in their quantum invariants. They are however distinguished by the next highest order coefficient, for $p \neq 3$, and therefore by the finite type invariant τ_p^{n+1}.

(To see this, one can work with the renormalized invariant $u_o\tau_p/h^n$, where u_o is the unit b_o/h^{2n} of Proposition 3.11, which assumes the value $\sum_{j\le k} q^{\mp 2j(j-1)}$ on $M_3^{\pm 2}$. The constant coefficients in these sums are both equal to $\sum_{j\le k} 1 = m(m+1)/2$, while the linear coefficients $\sum_{j\le k} \mp 8j(j-1) = \mp(m-1)m(m+1)(m+2)/3$ are distinct, since they are negatives and prime to p; see Remark 3.12c.) Therefore M_3^2 is *chiral*, that is has no orientation reversing automorphisms. A similar argument using Remark 3.12b shows that the manifold M_2, obtained by zero surgery on the Whitehead link, is chiral (also proved easily using Lescop's generalization of the Casson invariant).

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