MATH 542 FALL 2008 HOMEWORK PROBLEM SET 2

These are due in class next Thursday 9/11. You may work together. Then write them up on your own.

(1) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is given by $f(x, y, z) = (x^2 + y^2 - 4)^2 + z^2 - 1$. Calculate the Jacobian of f at a general point (x, y, z). At what points is f immersive? Submersive? Prove that $f^{-1}(0)$ is a compact surface without boundary. Food for thought: It might be fun to have Mathematica draw it to see what the genus is. Is there a systematic way to find the genus? I don't know but Dr. Hassett does!

(2) Suppose $f : M \to N$ is smooth and M is compact. Prove that the set of regular values of f is an open set. Give an example of such a map with an isolated critical point and isolated critical value.

(3) A tangent vector at $x \in M$ is sometimes defined as an equivalence class, $[\gamma]$, of smooth maps $\gamma : [0, a) \to M$ where $\gamma(0) = x$ and $\gamma \sim \omega$ if $\gamma(0) = \omega(0) = x$ and for some chart (U, ϕ) at x, the Jacobian matrix of $\phi \circ \gamma$ at 0 agrees with that of $\phi \circ \omega$. Show that this agreement is independent of choice of chart. Given such a definition of tangent space at x give a way to define a bijection to $T_x(M)$ as we defined it in class.

(4) Suppose $t \mapsto X(t) = (x(t), y(t))$ is an immersion of the circle into $\mathbb{R}^2$ (just imagine a smooth map of the interval [0,1] that happens to close up). What does mean about the velocity vector of X? Associate to X the map $S^1 \to S^1$ given by $t \to X'(t)/\|X'(t)\|$, and thus define an integer **index** or "winding number" to X. This is not the same as the winding number of the original X about the origin. Draw pictures of immersions of index zero, one and two. Argue that this integer is invariant under smooth homotopy of immersions (all intermediate maps are immersions). This last definition is a bit sloppy so your answer can also be sloppy. Is there anything special about the index of an embedding? Try to deform ones of different indices and see where it fails to be an immersion.