THE GROWTH RATE OF THE FIRST BETTI NUMBER IN ABELIAN COVERS OF 3-MANIFOLDS

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Abstract

We give examples of closed hyperbolic 3-manifolds with first Betti number 2 and 3 for which no sequence of finite abelian covering spaces increases the first Betti number. For 3-manifolds M with first Betti number 2 we give a characterization in terms of some generalized self-linking numbers of M, for there to exist a family of $\mathbb{Z}_n$ covering spaces, M_n, in which $\beta_1(M_n)$ increases linearly with n. The latter generalizes work of M. Katz and C. Lescop [KL], by showing that the non-vanishing of any one of these invariants of M is sufficient to guarantee certain optimal systolic inequalities for M (by work of Ivanov and Katz [IK]).

Introduction

Motivated by the attempt to classify all 3-dimensional manifolds via the **Geometrization Conjecture** of W. Thurston, it has been variously conjectured that, if M is an orientable, irreducible closed 3-manifold with infinite fundamental group, then:

Virtual Haken Conjecture. (VHC) M is finitely covered by a Haken manifold;

Virtual Positive Betti Number Conjecture. (VPBNC) Some finite cover of M has positive first Betti number;

Virtual Infinite Betti Number Conjecture. (VIBNC) Either $\pi_1(M)$ is virtually solvable or M has finite covers with arbitrarily large first Betti number;

Virtual Fibering Conjecture. (VFC) M has a finite cover that fibers over the circle.

There are easy implications VIBNC$\Longrightarrow$VPBNC$\Longrightarrow$VHC and VFC$\Longrightarrow$VPBNC $\Longrightarrow$VHC. Each implies, if M is atoroidal, the long-standing conjecture of Thurston that such a manifold admits a geometric structure. It is interesting to note that even if M is **assumed** to be hyperbolic, the conjectures above are open.

In this paper, we restrict our attention to VIBNC. (We note in passing that the alternative "$\pi_1(M)$ is virtually solvable" is sometimes replaced by the a priori stronger alternative that "M is finitely covered by the 3-torus, a nilmanifold or a solvmanifold.") One rich source of finite covering spaces are those obtained as iterated (regular) finite **abelian** covering spaces. Thus specifically, in this paper we consider the stronger:

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Question A. Does there exist an integer m, such that, if M is any closed, atoroidal 3-manifold with $\beta_1(M) \geq m$ then $\beta_1(M)$ can be increased in a finite abelian covering space?

Note that some condition on $H_1(M)$ is necessary, for if $H_1(M) = 0$, then M admits no non-trivial abelian covering spaces. Counter-examples also exist for many manifolds with $\beta_1(M) = 1$. For if M is zero-framed surgery on a knot in S^3, then it is easy to show that $H_1(\widetilde{M}; \mathbf{Q}) \cong \mathbf{Q} \oplus Q[t, t^{-1}]/\langle \Delta_k, t^n - 1\rangle$ where $\widetilde{M}$ is the n-fold cyclic cover and Δ_k is the Alexander polynomial of K. Thus $\beta_1(\widetilde{M}) = \beta_1(M) = 1$ except when Δ_k has a cyclotomic factor. We begin this paper by observing that counter-examples also exist in the cases $\beta_1(M) = 2$ and $\beta_1(M) = 3$.

Theorem. There exist closed hyperbolic 3-manifolds M with $\beta_1(M) = 2$ (respectively 3) for which no sequence of finite abelian covers increases the first Betti number.

It is noteworthy that Question A is still open.

If $\beta_1 > 0$, then there is an epimorphism $\pi_1(M) \to \mathbb{Z}$, and a corresponding sequence of finite cyclic covers of M. Our second contribution is, in the case $\beta_1(M) = 2$, to give necessary and sufficient conditions, of a somewhat geometric flavor, for the Betti number of these covers to increase linearly with the covering degree. This is the content of Section 2.

1. On abelian covers of hyperbolic 3-manifolds with $\beta_1(M) = 2$ and 3

In this section, we observe that, if Question A has an affirmative answer, then the integer m must be at least 4.

Theorem 1.1. *There exist closed hyperbolic 3-manifolds M with $\beta_1(M) = 2$ (respectively 3) such that if $\widetilde{M}$ is obtained from M by taking a sequence of finite abelian covering spaces, then $\beta_1(\widetilde{M}) = \beta_1(M)$.*

Proof. Begin with a "seed" manifold N whose fundamental group is nilpotent. If $\beta_1(N) = 2$ we let N be 0-framed surgery on the Whitehead link, commonly known as the **Heisenberg manifold**. The latter may also be described as the circle bundle over the torus with Euler class 1. It is well known that the fundamental group of this manifold is F/F_3 where F is the free group of rank 2 and F_3 is the third term of the lower-central series of F. If $\beta_1(N) = 3$, let $N = S^1 \times S^1 \times S^1$.

Next alter the seed manifold in a subtle way using the following result of A. Kawauchi [Ka; Corollary 4.3] (see also Boileau-Wang [BW section 4]).

Proposition 1.2. *(Kawauchi) For any closed 3-manifold N, there exists a hyperbolic 3-manifold M and a degree 1 map $f : M \to N$ that induces an isomorphism on homology groups with local coefficients in $\pi_1(N)$. In particular, if $\widetilde{N}$ is any regular covering space of N and $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ is the pull-back, then $\widetilde{f}$ induces isomorphisms on homology groups.*

Proof. To the best of our knowledge, this result was first established by Kawauchi using his theory of **almost identical imitations**. We sketch a proof using the approach of Boileau and Wang. Recall that any 3-manifold N contains a knot J whose exterior is hyperbolic. With more work, Boileau and Wang ensure that there exists such a knot J which is "totally null-homotopic", i.e., bounds a map of a 2-disk, $\phi : D^2 \to N$, such that the inclusion map $\pi_1(\text{image}\,\phi) \to \pi_1(N)$ is trivial. Let

M_n be the result of $1/n$-Dehn surgery on N along J. By work of W. Thurston, for almost all n, M_n is hyperbolic. Choose such an M_n and denote it by M. Since J is null-homotopic there is a degree one map $f : M \to N$ that induces an isomorphism on H_1.

Let $\widetilde{N}$ be a cover of N. Since J is null-homotopic, it lifts to $\widetilde{N}$, and there is an induced cover $\widetilde{M}$ and an induced map $\tilde{f} : \widetilde{M} \to \widetilde{N}$. Since J is totally null-homotopic, the pre-images of J bound disjoint Seifert surfaces in $\widetilde{M}$, and so $\tilde{f} : \widetilde{M} \to \widetilde{N}$ is an isomorphism on homology.

□

It follows from Proposition 1.2 that, for **any** covering space $\widetilde{M}$ of M that is "induced" from a cover $\widetilde{N}$ of N, the induced map $\tilde{f} : \widetilde{M} \to \widetilde{N}$ is an isomorphism on homology, so $\beta_1(\widetilde{M}) = \beta_1(\widetilde{N})$.

Specifically, letting $\widetilde{N}$ be the universal cover, $H_1(\widetilde{M}) \cong H_1(\widetilde{N}) = 0$ showing that that $\pi_1(\widetilde{M})$ is a perfect group. But $\pi_1(\widetilde{M})$ is kernel(f_*). Indeed, the condition that $f : M \to N$ induce an isomorphism on first homology with local coefficients in $\pi_1(N)$ is equivalent to the condition that the kernel of $f_* : \pi_1(M) \to \pi_1(N)$ be a perfect group.

Returning to the proof of our theorem, we claim that the manifold M satisfies the conclusion of the theorem. For suppose $\widetilde{M} \xrightarrow{p} M$ is a finite abelian covering space of M corresponding to a surjection $\psi : \pi_1(M) \to A$ where A is a finite abelian group. Since $f : M \to N$ induces an isomorphism on H_1, ψ factors through $f_* : \pi_1(M) \to \pi_1(N)$ via a surjection $\phi : \pi_1(N) \to A$. Therefore there is a finite abelian cover $\widetilde{N}$ of N and a lift $\tilde{f} : \widetilde{M} \to \widetilde{N}$. Thus by Proposition 1.2 $H_1(\widetilde{M}) \cong H_1(\widetilde{N})$. But we claim that $\beta_1(\widetilde{N}) \cong \beta_1(N)$. When N is $S^1 \times S^1 \times S^1$ this is clear. If N is the Heisenberg manifold, then first note that N is the circle bundle over a torus T with Euler class 1. The fiber is null-homologous so ϕ factors through $\pi_1(T)$. Thus there is a finite cover $g : \widetilde{T} \to T$ such that $\widetilde{N}$ is the circle bundle over the torus $\widetilde{T}$ obtained by pulling back N. Its Euler class, e, is equal to the degree of g, in particular is not zero. The fundamental group of $\widetilde{N}$ is $\langle x, y, t : [x, y] = t^e, [x, t], [y, t] \rangle$ and from this we see that $\beta_1(\widetilde{N}) = 2 = \beta_1(N)$. Thus $\beta_1(\widetilde{M}) = \beta_1(\widetilde{N}) = \beta_1(N) = \beta_1(M)$ as claimed. This shows that the first Betti number of M cannot be increased by a *single* abelian cover.

By Proposition 1.2 and an easy induction, we see that any *sequence* of abelian covers of M must be induced from a sequence of abelian covers of N. To finish the proof we only need to observe that $\beta_1 N$ cannot be increased by any sequence of abelian covers. Again, in the case where $N \cong S^1 \times S^1 \times S^1$, this is clear. If N is the Heisenberg manifold, note that our special argument above works as well for any bundle whose Euler class is merely non-zero (the fiber is still rationally null-homologous). Thus the iterated abelian covers of N continue to be circle bundles with non-zero Euler class, and the arguments of the previous paragraph apply to show that these covers all have $\beta_1 = 2$.

□

2. Linear Growth of Betti Numbers in Cyclic Covering Spaces

In this section we ask whether or not it is possible to increase the first Betti number with *linear growth rate* in some *compatible family* of cyclic covering spaces.

If M_∞ is a fixed infinite cyclic covering space corresponding to an epimorphism $\psi : \pi_1(M) \to \mathbb{Z}$ then by a *compatible family* we mean the usual family of finite cyclic covers M_n associated to $\pi_1(M) \to \mathbb{Z} \to \mathbb{Z}_n$. By a *linear growth rate* we mean $\varinjlim(\beta_1(M_n)/n)$ is positive. It already known that linear growth occurs precisely when $H_1(M_\infty)$ has positive rank as a $\mathbb{Z}[t, t^{-1}]$-module [Lu, Theorem 0.1][Lu, pg.35 Lemma 1.34,pg.453]. Therefore our contribution is to offer a more geometric way of viewing this criterion. We also point out an application to certain optimal systolic inequalities for such 3-manifolds as have appeared in work of Katz [IK][KL].

One should note from the outset that if $\pi_1(M)$ admits an epimorphism to $\mathbb{Z} * \mathbb{Z}$, then it is an easy exercise to elementary show that $\beta_1(M)$ can be increased linearly in finite cyclic covers since the same is patently true of the wedge of two circles. Such manifolds arise, for example, as 0-framed surgery on 2-component boundary links. This condition is not necessary however as we shall see in Example 2 below.

Suppose M is a closed, oriented 3-manifold with $\beta_1(M) = 2$. Given any basis $\{x, y\}$ of $H^1(M, \mathbf{Z})$ we shall define a sequence of higher-order invariants $\beta^n(x, y)$; $n \geq 1$ taking values in sets of rational numbers. The invariants can be interpreted as certain Massey products in M. The invariant $\beta^1(x, y)$ is always defined, is independent of basis, and essentially coincides with the invariant λ, an extension of Casson's invariant, due to Christine Lescop [Les]. If β^i is defined for all $i < n$ and is zero, then β^n is defined (this is why the invariants are called higher-order). If $H_1(M)$ has no torsion, so that M can be viewed as 0-framed surgery on a 2-component link in a homology sphere (with Seifert surfaces dual to $\{x, y\}$) then β^n, when defined, is the same as the sequence of link concordance invariants of the same name due to the first author [C1]. In this case β^1 was previously known as the Sato-Levine invariant.

After defining the invariants $\beta^n(x, y)$, we show that their vanishing is equivalent to the linear growth of Betti numbers in the family corresponding to the infinite cyclic cover associated to x.

Theorem 2.1. *Let M be a closed oriented 3-manifold with $\beta_1(M) = 2$. The following are equivalent.*

- A. *There exists a compatible family $\{M_n | n \geq 1\}$ of finite cyclic covers of M such that $\beta_1(M_n)$ grows linearly with n.*
- B. *There exists a primitive class $x \in H^1(M; \mathbf{Z})$ such that for **any** basis $\{x, y\}$ of $H^1(M; \mathbf{Z})$, $\beta^n(x, y) = 0$ for all $n \geq 1$.*
- C. *There exists a primitive class $x \in H^1(M; \mathbf{Z})$ such that for **some** basis $\{x, y\}$ of $H^1(M; \mathbf{Z})$, $\beta^n(x, y)$ can be defined and contains 0 for each $n \geq 1$.*

Corollary 2.2. *Let M be a closed oriented 3-manifold with $\beta_1(M) = 2$. Let $\widetilde{M}$ denote the universal torsion-free abelian ($\mathbf{Z} \oplus \mathbf{Z}$) cover of M. Let $[F]$ denote the class in $H_1(\widetilde{M})$ of a lift of a typical fiber of the Abel-Jacobi map of M (represented by a lift of the circle we called $c(x, y)$ above). If, for **some** $\{X, Y\}$, and **some** n, $\beta^n(X, Y) \neq 0$ then $[F]$ is non-zero.*

The above Corollary generalizes an (independent) result of A. Marin (see Prop.12.1 of [KL]), which dealt with only the case $n = 1$. The significance of this Corollary is that it has been previously shown by Ivanov and Katz ([IK, Theorem 9.2 and Cor.9.3]) that the conclusion of Corollary 2.2 is sufficient to guarantee a certain optimal systolic inequality for M. The interested reader is referred to those works.

Suppose c and d are disjointly embedded oriented circles in M that are zero in $H_1(M;\mathbf{Q})$. Then the **linking number of c with d**, $\ell k(c,d) \in \mathbf{Q}$ is defined as follows. Choose an embedded oriented surface V_d whose boundary is "m times d" (i.e. a circle in a regular neighborhood N of d that is homotopic in N to md) for some positive integer m, and set:

$$\ell k(c,d) = \frac{1}{m}(V_d \cdot c).$$

Given this, the invariants $\beta^n(x,y)$ are defined as follows. Let $\{V_x, V_y\}$ be embedded, oriented connected surfaces that are Poincaré Dual to $\{x,y\}$ and meet transversely in an oriented circle that we call $c(x,y)$ (by the proof of [C1, Theorem 4.1] we may assume that $c(x,y)$ is connected). Let $c^+(x,y)$ denote a parallel of $c(x,y)$ in the direction given by V_y. Note that $\{V_x, V_y\}$ induce two maps ψ_x, ψ_y from M to S^1 wherein the surfaces arise as inverse images of a regular value. The product of these maps yields a map $\psi : M \to S^1 \times S^1$ that induces an isomorphism on H_1/torsion. Since $c(x,y)$ and $c^+(x,y)$ are mapped to points under ψ, they represent the zero class in $H_1(M;\mathbf{Q})$. Therefore we may define $\beta^1(x,y) = \ell k(c(x,y),\ c^+(x,y))$. In fact, $-\beta^1(x,y) \cdot |\operatorname{Tor} H_1(M;\mathbf{Z})|$ is precisely Lescop's invariant of M [Les; p.90-94]. An example is shown in Figure 1 of a manifold with $\beta^1(x,y) = -k$.

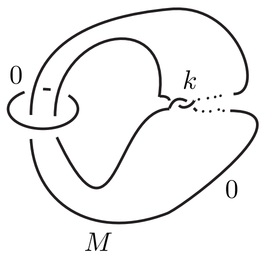

FIGURE 1. Example of $\beta^1(x,y) = -k$

The idea of the higher invariants is to iterate this process as long as possible (compare [C1]). Since $c^+(x,y))$ is rationally null-homologous, there is a surface $V_{c(x,y)}$ whose boundary is "k times $c^+(x,y))$" (in the sense above). We could then define $c(x,x,y)$ to be $V_x \cap V_{c(x,y)}$, an embedded oriented circle on V_x. If $c(x,x,y)$ is rationally null-homologous, then $\beta^2(x,y)$ is defined as $\ell k(c(x,x,y), c^+(x,x,y))$ and we may also continue and define $c(x,x,x,y)$. In general $c(x,x\ldots,x,y) = c(x^n,y)$ will be able to be defined using the chosen surfaces if $c(x^{n-1},y)$ is defined and is also rationally null-homologous (but to do so involves one more choice of a bounding surface). Once $c(x^n,y)$ is defined and is rationally null-homologous, we may define $\beta^n(x,y)$. In general, we do not claim that the value of $\beta^n(x,y)$ is independent of the choices of surfaces. Therefore the invariants can be thought of as taking values in a set, just like Massey products. This indeterminacy will not concern us here, for we are only interested in the first non-vanishing value (if it exists) and we shall see that this is independent of the surfaces.

Much of the time it is convenient to abbreviate $c(\overbrace{x\ldots x}^{n},y)$ as $c(n)$ so $c(x,y) = c(1)$.

Definition. If $c(n)$ is defined and rationally null-homologous then $\beta^n(x,y)$ is defined to be the set of rational numbers $\ell k(c(n), c^+(n))$, ranging over all possible ways of defining such a $c(n)$. If no such $c(n)$ exists then $\beta^n(x,y)$ is undefined.

Example 1. *Consider the manifold M, shown in Figure 2, obtained from 0-framed surgery on a two component link $\{L_x, L_y\}$. Use a genus one Seifert surface for L_y obtained from the obvious twice-punctured disk and a tube that goes up to avoid L_x. Let V_y be this surface capped-off in M. Similarly use the fairly obvious Seifert surface for L_x in the complement of L_y. Then $c^+(x,y)$ is shown. Since it has self-linking zero with respect to V_x, $\beta^1(x,y) = \beta^1(y,x) = 0$. Furthermore $\beta^2(x,y) = -1$ (note the link $\{c^+(x,y), L_x\}$ is very similar to that of Figure 1). This means that $\pi_1(M)$ does* ***not*** *admit an epimorphism to $\mathbb{Z} * \mathbb{Z}$ since that would imply that $\{L_x, L_y\}$ were a homology boundary link. But $\beta^2(x,y) = -1$ precludes this by* [C2]. *Nonetheless, further $c(yy...y,x)$ may be taken to be empty since $c^+(x,y)$ and L_y form a boundary link in the complement of L_x. Thus $\beta^n(y,x) = 0$ for all n, indicating, by Theorem 2.1, that the first Betti numbers will grow linearly in the family of finite cyclic covers corresponding to the map $\pi_1(M) \to \mathbb{Z}$ that sends a meridian of L_x to zero and a meridian of L_y to one.*

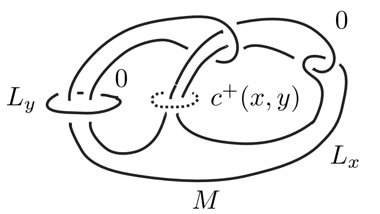

FIGURE 2. Example with linear growth in cyclic covers but no map to $\mathbb{Z} * \mathbb{Z}$

Example 2. *Consider the family of manifolds M_k, shown in Figure 3 and Figure 4, obtained from 0-framed surgery on a two component link.*

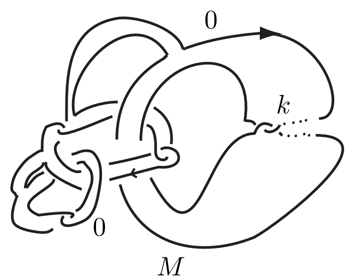

FIGURE 3. Example with $\beta^1(x,y) = 0$,$\beta^2(x,y) = -k$, $\beta^2(y,x) = -1$

If V_x denotes the capped-off Seifert surface (obtained using Seifert's algorithm) for the link component, L_x, on the right-hand side and V_y denotes the capped-off Seifert surface for the link component, L_y, on the left-hand side, then the dashed circle in Figure 4 is $c(x,y) = V_x \cap V_y$. The circle $c(y,x)$ is merely this circle with opposite orientation. Since it lies on an untwisted band of V_x, $\beta^1(x,y) = 0 =$

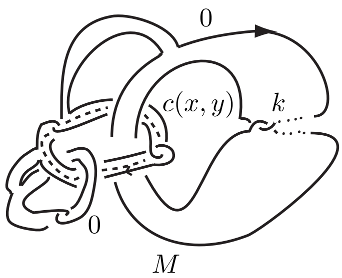

FIGURE 4. The circle $c(x, y)$

$\beta^1(y, x)$. Therefore the Lescop invariant of M vanishes. But the link $\{c(x, y), L_x\}$ is the link of Figure 1 so $\beta^2(x, y) = -k$, whereas the link $\{c(y, x), L_y\}$ is a Whitehead link so $\beta^2(y, x) = -1$. We claim further that, as long as $k \neq 0$, for ***any*** *basis $\{X, Y\}$ of $H^1(M)$, $\beta^2(X, Y) \neq 0$. It will then follow from Theorem 2.1 that the first Betti number of M will grow sub-linearly in* ***any*** *family of finite cyclic covers. A general basis, $\{V_X, V_Y\}$, of $H_2(M)$ can be represented as follows. Represent V_X by p parallel copies of V_x together with q parallel copies V_y, and represent V_Y by r parallel copies of V_x together with s parallel copies V_y, where $ps - qr = \pm 1$. Thus $c(X, Y) = -c(Y, X)$ is represented by $ps - qr$ parallel copies of $c(x, y)$. It follows that $\beta^1(X, Y) = \beta^1(x, y) = 0$, reinforcing our above claim that β^1 is independent of basis. Hence $V_{c(X,Y)} = \pm V_{c(x,y)}$ so $c(X, X, Y)$ is represented by $\pm pc(x, x, y) \mp qc(y, y, x)$. Since $\beta^2(X, Y)$ is the self-linking number of this class, it can be evaluated to be*

$$p^2\beta^2(x, y) + q^2\beta^2(y, x) - 2pq\ell k(c(x, x, y), c(y, y, x))$$

but the latter mixed linking number is easily seen to be zero in this case. Hence $\beta^2(X, Y) = -kp^2 - q^2$ which is non-zero if k is non-zero.

Lemma 2.3. *Suppose $c(1), \dots, c(n)$ have been defined as embedded oriented curves on V_x arising as $c(1) = V_x \cap V_y$ and*

$$c(j) = V_x \cap V_{c(j-1)} \qquad 2 \leq j \leq n$$

where $V_{c(j)}$, $1 \leq j \leq n-1$, is a embedded, oriented connected surface whose boundary is a positive multiple k_j of $c^+(j)$ (in the sense above). Then β^j is defined for $1 \leq j \leq n-1$ and the following are equivalent:

- *B1: $\beta^1, \dots, \beta^n$ are defined using the given system of surfaces.*
- *B2: β^j is defined for $1 \leq j \leq n$ and is* **zero** *for $1 \leq j \leq \left[\frac{n}{2}\right]$*
- *B3: $c(n+1)$ exists*
- *B4: For all s, t such that $1 \leq s \leq t$ and $s + t \leq n$, $\ell k(c(s), c^+(t)) = 0$.*

Proof of Proposition 2.3. Assume $1 \leq j \leq n-1$. The hypotheses imply that a positive multiple of $c^+(j)$ is (homotopic to) the boundary of a surface so $c^+(j)$ and $c(j)$ are rationally null-homologous. Thus their linking number is well-defined, establishing the first claim.

B1$\Longleftrightarrow$B3: β^n is defined precisely when $[c(n)] = 0$ in $H_1(M; \mathbf{Q})$ which is precisely the condition under which $c(n+1)$ can be defined.

B1$\Longrightarrow$B4: If $n = 1$ the implication is true since B4 is vacuous. Thus assume by induction that the implication is true for $n-1$, that is our inductive assumption is that, for all $s+t<n$, $\ell k(c(s), c^+(t)) = 0$. Now consider the case that $s+t=n$. Since β^n is defined $[c(n)] = 0$ in $H_1(M;\mathbf{Q})$. We claim this is true precisely when $c(n)\cdot c(1) = 0$ (**here we refer to oriented intersection number on the surface** V_x). For suppose $\psi_x : M \to S^1$ and $\psi_y : M \to S^1$ are maps such that $\psi_x^{-1}(*) = V_x$ and $\psi_y^{-1}(*) = V_y$. Then $(\psi_x)_*([c(n)]) = 0$ since $c(n) \subset V_x$; and $(\psi_y)_*([c(n)]) = 0$ precisely when $c(n)\cdot V_y = c(n)\cdot c(1) = 0$. But the map $\psi_x \times \psi_y$ completely detects $H_1(M)/\text{Torsion}$. Therefore, once $c(n)$ exists, β^n is defined if and only if:

$$\begin{aligned} 0 = c(n)\cdot c(1) &= \pm(c(1)\cdot V_{c(n-1)}) \\ &= \pm k_{n-1}\ell k(c(1), c^+(n-1)) \end{aligned}$$

which establishes B4 in the case $s=1$. But we claim that, if B4 is true for $s+t<n$, then for $s+t=n$ and $s<t$,

$$k_{t-1}\ell k(c(s+1), c^+(t-1)) = k_s \ell k(c(s), c^+(t)).$$

This equality can then be applied, successively decreasing s, to establish B4 in generality. This claimed equality is established as follows.

$$\begin{aligned} \pm k_{t-1}\ell k(c(s+1), c^+(t-1)) &= \pm V_{c(t-1)}\cdot c(s+1) \\ &= \pm V_{c(t-1)}\cdot (V_x \cap V_{c(s)}) \\ &= V_{c(s)}\cdot (V_x \cap V_{c(t-1)}) \\ &= V_{c(s)}\cdot c(t) \\ &= k_s \ell k(c(t), c^+(s)) \\ &= k_s \ell k(c(s), c^+(t)). \end{aligned}$$

The last step is justified by verifying that $c(s)\cdot c(t) = 0$ if $s<t$. For

$$\begin{aligned} c(s)\cdot c(t) &= c(s)\cdot V_{c(t-1)} \\ &= \pm k_{t-1}\ell k(c(s), c^+(t-1)) \end{aligned}$$

which vanishes by our inductive assumption since $s+(t-1)<n$.

B4$\Longrightarrow$B1: Since β^1 is always defined we may assume $n>1$. It follows from B4 that $\ell k(c(1), c^+(n-1)) = 0$ if $n>1$. But we saw in the proof of B1$\Longrightarrow$B4 that once $c(n)$ was defined, this was equivalent to β^n being defined.

B2$\Longrightarrow$B1: This is obvious.

B4$\Longrightarrow$B2: Since B4$\Longrightarrow$B1, we have β^j defined for $j \le n$. Now suppose $1 \le j \le \left[\frac{n}{2}\right]$. Since $\beta^j = \ell k(c(j), c^+(j)$ and $2j \le n$, this vanishes by B4.

This completes the proof of Lemma 2.3. $\square$

Proof of Theorem 2.1. The proof shows slightly more, namely that there is a correspondence between the infinite cyclic cover implicit in part A and the class x in parts B and C. Suppose $\{M_n\}$ is a family of n-fold cyclic covers of M corresponding to the infinite cyclic cover M_∞. Note that $H_1(M_\infty;\mathbf{Q})$ is a finitely generated $\Lambda = \mathbf{Q}[t,t^{-1}]$ module (this involves a choice of generator of the infinite cyclic group of deck translations of M_∞). Throughout this proof, homology will be taken with rational coefficients unless specified otherwise.

Step 1: $\beta_1(M_n)$ grows linearly $\Longleftrightarrow$ $H_1(M_\infty;\mathbf{Q})$ has positive rank as a Λ-module.

This fact is well-known (see for example). We present a quick proof for the convenience of the reader. We are indebted to Shelly Harvey for showing us this elementary proof. Since Λ is a PID,

$$H_1(M_\infty) \cong \Lambda^{r_1} \oplus_j \frac{\Lambda}{\langle p_j(t)\rangle}$$

where $p_j(t) \neq 0$. By examining the "Wang sequence" with **Q**-coefficients

$$H_2(M_\infty) \longrightarrow H_2(M_n) \xrightarrow{\partial_*} H_1(M_\infty) \xrightarrow{t^n-1} H_1(M_\infty) \xrightarrow{\pi} H_1(M_n) \xrightarrow{\partial_*} H_0(M_\infty) \xrightarrow{t^n-1}$$

it is easily seen that

$$\begin{aligned} H_1(M_n) &\cong \frac{H_1(M_\infty)}{\langle t^n-1\rangle} \oplus \mathbf{Q} \\ &\cong \left(\frac{\Lambda}{\langle t^n-1\rangle}\right)^{r_1} \oplus_j \frac{\Lambda}{\langle p_j(t), t^n-1\rangle} \oplus \mathbf{Q}. \end{aligned}$$

The first summand contributes nr_1 to $\beta_1(M_n)$. The **Q**-rank of the second summand is bounded above by the sum of the degrees of the p_j, a number that is **independent** of n. Therefore $\beta_1(M_n)$ grows linearly with n if $r_1 \neq 0$ and otherwise is bounded above by a constant (independent of n).

Step 2: $H_1(M_\infty)$ has positive Λ-rank $\Longleftrightarrow$ $H_1(M_\infty)$ has no $(t-1)$-torsion (equivalently $t-1$ acts injectively). To verify Step 2, consider the "Wang sequence" with **Q**-coefficients

$$H_2(M_\infty) \longrightarrow H_2(M) \xrightarrow{\partial_*} H_1(M_\infty) \xrightarrow{t-1} H_1(M_\infty) \xrightarrow{\pi} H_1(M) \xrightarrow{\partial_*} H_0(M_\infty) \xrightarrow{t-1} H_0(M_\infty)$$

associated to the exact sequence of chain complexes

$$0 \longrightarrow C_*(M_\infty;\mathbf{Q}) \xrightarrow{t-1} C_*(M_\infty;\mathbf{Q}) \xrightarrow{\pi} C_*(M;\mathbf{Q}) \longrightarrow 0.$$

Since $H_0(M_\infty) \cong \mathbf{Q}$, image $\partial_* \cong \mathbf{Q}$ on $H_1(M)$. If $\beta_1(M) = 2$ then it follows that $\mathbf{Q} \cong \ker \partial_* = \text{image}(\pi) \cong \text{cokernel}(t-1)$. It follows that $H_1(M_\infty)$ contains at most one summand of the form $\Lambda/\langle (t-1)^m\rangle$ since each such summand contributes precisely one **Q** to cokernel$(t-1)$. Similarly each Λ summand of $H_1(M_\infty)$ contributes one **Q** to the cokernel. Therefore $H_1(M_\infty)$ has positive Λ rank if and only if it has no summand of the form $\Lambda/\langle (t-1)^m\rangle$. The latter is equivalent to saying that it has no $(t-1)$-torsion, or that $t-1$ acting on $H_1(M_\infty)$ is injective. This completes Step 2.

Step 3: $(t-1) : H_1(M_\infty) \to H_1(M_\infty)$ is injective $\Longleftrightarrow$ For any surface V_x, dual to x, and for each surface V_y such that $\{[V_y],[V_x]\}$ generates $H_2(M;\mathbf{Z})$, the class $[\tilde{c}(x,y)] \in H_1(M_\infty;\mathbf{Q})$ is zero. Moreover the latter statement is equivalent to one where "for each" is replaced by "for some".

Suppose that $t-1$ is injective. Note that the injectivity of $t-1$ is equivalent to $\partial_* : H_2(M) \to H_1(M_\infty)$ being the zero map. Then, for **any** $[V_y]$ as above, $\partial_*([V_y]) = 0$. But we claim that $\partial_*([V_y])$ is represented by $[\tilde{c}(x,y)]$, since V_x is Poincaré Dual to the class x defining M_∞. For if $Y = M - \text{int}(V_x \times [-1,1])$ then a copy of Y, denoted $\widetilde{Y}$, can be viewed as a fundamental domain in M_∞, as shown in Figure 5.

Moreover if $\widetilde{V}_y$ denotes $p^{-1}(V_y) \cap \widetilde{Y}$ then $\widetilde{V}_y$ is a compact surface in $\widetilde{M}$ whose boundary is $t_*(\tilde{c}(x,y)) - \tilde{c}(x,y)$. Thus $\widetilde{V}_y$ is a 2-chain in M_∞ such that $\pi_\#(\widetilde{V}_y)$ gives the chain representing $[V_y]$. Since $\partial\widetilde{V}_y$ is $(t-1)\tilde{c}(x,y)$ in $C_*(M_\infty;\mathbf{Q})$, it follows from

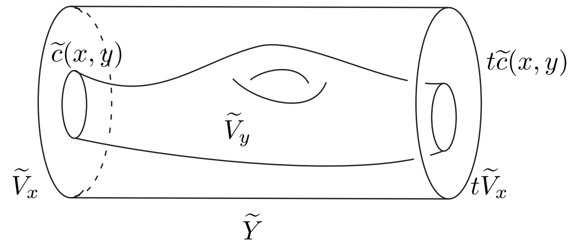

FIGURE 5. Fundamental Domain of M_∞

the explicit construction of ∂_* in the proof of the Zig-Zag Lemma [Mu,Section 24] that $\partial_*([V_y]) = [\tilde{c}(x,y)]$.

Conversely, if $\partial_*([V_y]) = 0$ for **some** $[V_y]$ then ∂_* is the zero map (note that since V_x lifts to M_∞, $[V_x]$ lies in the image of $H_2(M_\infty) \longrightarrow H_2(M)$ so $\partial_*([V_x]) = 0$ t).

Therefore the injectivity statement implies the "for each" statement which clearly implies the "for some" statement. Conversely, the "for some" statement implies the injectivity statement.

Step 4: The class $[\tilde{c}(x,y)]$ from Step 3 is 0 if and only if it is divisible by $(t-1)^k$ for every positive k. In fact it suffices that it be divisible by $(t-1)^N$ where N is the largest nonnegative integer such that $\Lambda/\left\langle (t-1)^N \right\rangle$ is a summand of $H_1(M_\infty, \mathbf{Q})$.

One implication is immediate, so assume that there exists a class $[V_y]$ as in Step 3 such that $\partial_*([V_y]) = [\tilde{c}_1] = (t-1)^N\beta$ for some $\beta \in H_1(M_\infty)$. Since $[\tilde{c}_1] \in \text{image}\,\partial_*$, it is $(t-1)$-torsion so β is $(t-1)^{N+1}$-torsion. Moreover β lies in the submodule $A \subset H_1(M_\infty, \mathbf{Q})$ consisting of elements annihilated by some power of $t-1$, so, by choice of N, $(t-1)^N\beta = 0 = [\tilde{c}_1] = 0$ as desired. This completes the verification of Step 4.

Step 5: C⇒A Let $\{x, y\}$ be as in the hypotheses of C and let M_∞ correspond to the class x. Let N be the positive integer as above. If $\beta^{(N+1}$ can be defined, we know in particular that there exists some system of surfaces $\{V_x, V_y, ..., V_{c(N)}\}$ that defines $\{c(j)\}$, $1 \le j \le (N+1)$. Choose a preferred lift $\widetilde{V}_x$, of V_x to M_∞ and a preferred fundamental domain $\widetilde{Y}$ as above lying on the positive side of $\widetilde{V}_x$. Consider any $m, 1 \le m \le N$. Since $c(m)$ and $c(m+1)$ lie on V_x, they lift to oriented 1-manifolds $\tilde{c}(m)$ and $\tilde{c}(m+1)$ in $\widetilde{V}_x$. Similarly $c^+(m)$ lifts to $\tilde{c}^+(m)$, which is a push-off of $\tilde{c}(m)$ lying in $\widetilde{Y}$. Recall that $c(m+1) = V_{c(m)} \cap V_x$ where $\partial V_{c(m)} = k_m c^+_{(m)}$ for some positive integer k_m. Letting $\widetilde{V}_{c(m)}$ be $V_{c(m)}$ cut open along $c(m+1)$ we observe that $\widetilde{V}_{c(m)}$ can be lifted to $\widetilde{Y}$ and viewed as a 2-chain showing that $k_m[\tilde{c}^+(m)] = (t-1)[\tilde{c}(m+1)]$ in $H_1(M_\infty; \mathbf{Q})$, as in Figure 6.

Thus $[\tilde{c}^+(1)] = (t-1)(1/k_1)[\tilde{c}(2)] = (t-1)^2(1/k_1)(1/k_2)[\tilde{c}(3)]$, et cetera, showing that $[\tilde{c}(1)]$ is divisible by $(t-1)^N$. By Steps 1 through 4, this implies A of Theorem 2.1, completing Step 5.

Step 6: A⇒B We assume that there is a primitive class $x \in H^1(M; \mathbf{Z})$ corresponding to M_∞ and $\{M_n\}$ where $\beta_1(M_n)$ grows linearly. By Steps 1, 2, and 3, for any $\{x, y\}$ generating $H^1(M; \mathbf{Z})$, $H_1(M_\infty; \mathbf{Q})$ has no $(t-1)$-torsion, and for any surfaces dual to x, y, $[\tilde{c}(1)] = 0$. Recall that $c(1)$ and $c(2)$ are always defined. We shall establish inductively that for all $m \ge 2$, $c(m)$ is defined and that for *any*

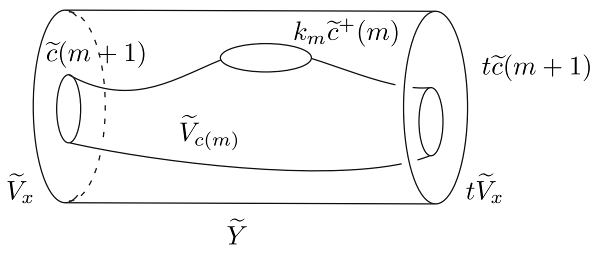

FIGURE 6

system of surfaces used to define $c(m-1)$, $[\tilde{c}(m-1)] = 0$ in $H_1(M_\infty; \mathbf{Q})$. This has already been shown for $m = 2$. Suppose it has been established for m (and all lesser values). We now establish it for $m+1$. Since $c(m)$ and $c(m-1)$ exist, the argument in Step 5 (Figure 6) shows that $k_{m-1}[\tilde{c}(m-1)] = (t-1)[\tilde{c}(m)]$ in $H_1(M_\infty; \mathbf{Q})$. But $[\tilde{c}(m-1)] = 0$ so $[\tilde{c}(m)]$ is $(t-1)$-torsion. Since there is no non-trivial $(t-1)$-torsion, $[\tilde{c}(m)] = 0$ in $H_1(M_\infty; \mathbf{Q})$. Hence $[c(m)] = 0$ in $H_1(M; \mathbf{Q})$ so $c(m+1)$ is defined. Since this holds for any system of defining surfaces, this completes the inductive step.

Since $c(m)$ is defined for all $m \geq 1$, by Lemma 2.3, $\beta^n(x, y) = 0$ for all $n \geq 1$. This completes the proof of Step 6.

Since B clearly implies C, this completes the proof of Theorem 2.1. □

Proof of Corollary 2.2. Assume some $\beta^m(x, y) \neq 0$. If $[F]$ were zero then certainly, for the fixed infinite cyclic cover, M_∞, corresponding to x, $[\widetilde{c}(x, y)] = 0$ in $H_1(M_\infty; \mathbb{Q})$ so by Steps $1-3$ of the above proof, $\beta_1(M_n)$ grows linearly with n. By Theorem 2.1, this would imply that $\beta^m(x, y) = 0$ for all m, a contradiction. □

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