

Extra Credit Assignment: Nested Radicals
Due Monday, 26 November 2007

Instructions: Complete as many questions as you can. You will receive one point for each correct answer. The questions are designed to be done in order, but when completing a question, you may assume the answers to any of the previous questions (for example, you may answer question 5 even though you may not have completed questions 1 through 4). It is suggested that you try the last four questions first, since they are designed to be more computational and may help your understanding of the first eight questions, which are more conceptual.

1. Let p and q be positive real numbers. Consider the following recursively defined sequence $\{a_n\}_{n=1}^{\infty}$:

$$a_1 = \sqrt{q} \quad \text{and} \quad a_{n+1} = \sqrt{q + pa_n} \quad \text{for } n \geq 1 \quad (1)$$

Write out a_1 , a_2 , a_3 , and a_4 . Explain why we might want to interpret the nested radical

$$\sqrt{q + p\sqrt{q + p\sqrt{q + p\sqrt{q + \cdots}}}} \quad (2)$$

as the limit of the sequence $\{a_n\}_{n=1}^{\infty}$.

2. We are going to show that $\{a_n\}_{n=1}^{\infty}$ is a bounded monotone sequence, and that, hence, it has a limit by the bounded monotone sequence property of the real numbers. First, show that, because p and q are positive, the two roots of the quadratic equation $A^2 - pA - q = 0$,

$$\frac{p - \sqrt{p^2 + 4q}}{2} \quad \text{and} \quad \frac{p + \sqrt{p^2 + 4q}}{2} \quad (3)$$

are real, and that one is negative and one is positive.

3. Now show that, because $a_1 = \sqrt{q}$, we have that

$$\frac{p - \sqrt{p^2 + 4q}}{2} \leq a_1 \leq \frac{p + \sqrt{p^2 + 4q}}{2} \quad (4)$$

The first inequality follows quickly from question 2; the second inequality is a bit harder.

4. Next, show that, if

$$a_n \leq \frac{p + \sqrt{p^2 + 4q}}{2} \quad (5)$$

then

$$a_{n+1} = \sqrt{q + pa_n} \leq \frac{p + \sqrt{p^2 + 4q}}{2} \quad (6)$$

One strategy would be to start with the first inequality and “build” $a_{n+1} = \sqrt{q + pa_n}$ by multiplying both sides by p , adding q to both sides, and taking the positive square root of both sides. It might be helpful in the last step, as a side calculation, to expand out the expression

$$\left(\frac{p + \sqrt{p^2 + 4q}}{2} \right)^2 \quad (7)$$

5. Based on questions 3 and 4, argue that $\frac{p + \sqrt{p^2 + 4q}}{2}$ is an upper bound for the sequence $\{a_n\}_{n=1}^{\infty}$.

This sort of argument is called a proof by induction. Give a much simpler reason why $\frac{p - \sqrt{p^2 + 4q}}{2}$ is a lower bound for the sequence $\{a_n\}_{n=1}^{\infty}$.

6. We now know that $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence. Now show that, because for all $n \geq 1$

$$\frac{p - \sqrt{p^2 + 4q}}{2} \leq a_n \leq \frac{p + \sqrt{p^2 + 4q}}{2} \quad (8)$$

we must have that

$$a_n \leq a_{n+1} = \sqrt{q + pa_n} \quad (9)$$

or all $n \geq 1$. You might want to try solving the inequality $A^2 - pA - q \leq 0$ first.

7. Now we know that $\{a_n\}_{n=1}^{\infty}$ is a bounded monotone sequence, and thus has a limit A . We also know that $\{a_n\}_{n=1}^{\infty}$ is a recursively defined sequence. Based on your knowledge of recursive sequences, show that, because $a_{n+1} = \sqrt{q + pa_n}$, the only two finite possibilities for A are

$$\frac{p - \sqrt{p^2 + 4q}}{2} \quad \text{and} \quad \frac{p + \sqrt{p^2 + 4q}}{2} \quad (10)$$

8. Finally, explain why questions 6 and 7 imply that

$$\lim_{n \rightarrow \infty} a_n = \sqrt{q + p\sqrt{q + p\sqrt{q + p\sqrt{q + \dots}}}} = \frac{p + \sqrt{p^2 + 4q}}{2} \quad (11)$$

9. Using the conclusion of question 8, find

$$\sqrt{42 + \sqrt{42 + \sqrt{42 + \sqrt{42 + \dots}}}} \quad (12)$$

10. Using question 9 as a possible guide, find a way to write any integer $k > 1$ as a nested radical of the form

$$\sqrt{m + \sqrt{m + \sqrt{m + \sqrt{m + \dots}}}} \quad (13)$$

where m is a positive integer.

11. Show that the golden ratio $\phi = \frac{1 + \sqrt{5}}{2} = 1.61803\dots$, which is ubiquitous in ancient Greek architecture, equals the nested radical

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}} \quad (14)$$

12. Is it ever possible for the sequence $\{a_n\}_{n=1}^{\infty}$ to be constant, given that p and q are both positive? Why or why not?