Dynamical Systems

Fall 2007 **Dr. D. Lenz**

Homework 1 **Due September 7, 2007**

(1) Let (X, d) be a complete metric space and $L : X \longrightarrow X$ a contraction with contracting factor $\vartheta < 1$ and fixed point x_F. Let $x \in X$ be arbitrary and define $x_0 := x$ and $x_n := L^n(x)$ for $n \geq 1$. Show the apriori error estimate

$$d(x_n, x_F) \leq \vartheta^n \frac{d(x_0, x_1)}{1 - \vartheta}.$$

Hint: $x_F = \lim_{m\to\infty} x_m$.

(2) Let $a > 0$, $C : [0, a] \longrightarrow [0, \infty)$ monotonously increasing and $K : [0, a] \longrightarrow [0, \infty)$ continuous. Show that any continuous $u : [0, a] \longrightarrow [0, \infty)$ with

$$0 \leq u(t) \leq C(t) + \int_0^t K(s)u(s)ds$$

satisfies the generalized Gronwall estimate

$$0 \leq u(t) \leq C(t) \exp(\int_0^t K(s)ds)$$

for any $t \in [0, a]$.

(3) Let $E \subset \mathbb{R}^n$ be open and $f \in C^1(E)$. Show that any solution $\varphi : I \longrightarrow E$ of $x' = f(x)$ is twice continuously differentiable.

(4) (a) Compute the first three successive approximants u_1, u_2 and u_3 for the initial value problem $x' = x^2$, $x(0) = 1$. Show by induction that $u_n(t) = 1 + t + \cdots + t^n + O(t^{n+1})$ as $t \to \infty$.

(b) Solve the initial value problem in (a) and write down the Taylor expansion of the solution. Hint: The solution should be $x(t) = 1/(1-t)$.