(1) Let \((X,d)\) be a complete metric space and \(L : X \to X\) a contraction with contracting factor \(\vartheta < 1\) and fixed point \(x_F\). Let \(x \in X\) be arbitrary and define \(x_0 := x\) and \(x_n := L^n(x)\) for \(n \geq 1\). Show the apriori error estimate
\[
d(x_n, x_F) \leq \vartheta^n \frac{d(x_0, x_1)}{1 - \vartheta}.
\]
Hint: \(x_F = \lim_{m \to \infty} x_m\).

(2) Let \(a > 0, C : [0, a] \to [0, \infty)\) monotonously increasing and \(K : [0, a] \to [0, \infty)\) continuous. Show that any continuous \(u : [0, a] \to [0, \infty)\) with
\[
0 \leq u(t) \leq C(t) + \int_0^t K(s)u(s)ds
\]
satisfies the generalized Gronwall estimate
\[
0 \leq u(t) \leq C(t) \exp\left(\int_0^t K(s)ds\right)
\]
for any \(t \in [0, a]\).

(3) Let \(E \subset \mathbb{R}^n\) be open and \(f \in C^1(E)\). Show that any solution \(\varphi : I \to E\) of \(x' = f(x)\) is twice continuously differentiable.

(4) (a) Compute the first three successive approximants \(u_1, u_2\) and \(u_3\) for the initial value problem \(x' = x^2, x(0) = 1\). Show by induction that \(u_n(t) = 1 + t + \cdots + t^n + O(t^{n+1})\) as \(t \to \infty\).
(b) Solve the initial value problem in (a) and write down the Taylor expansion of the solution. Hint: The solution should be \(x(t) = 1/(1-t)\).