Dynamical Systems

Fall 2007

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Homework 2 Due September 14, 2007

(1) Find the maximal interval of existence \((\alpha, \beta)\) for the following initial value problems and if \(\alpha > -\infty\) or \(\beta < \infty\) discuss the limit of the solution as \(t \to \alpha\) or as \(t \to \beta\):
   (a) \(x' = x^2, \ x(0) = x_0\).
   (b) \(x' = \sec x, \ x(0) = 0\).
   (c) \(x' = x^2 - 4, \ x(0) = 0\).

(2) Find the maximal interval of existence \((\alpha, \beta)\) for the following initial value problems and if \(\alpha > -\infty\) or \(\beta < \infty\) discuss the limit of the solution as \(t \to \alpha\) or as \(t \to \beta\):
   (a) \((x_1, x_2)' = (\frac{1}{2x_1}, x_2^2), \ (x_1(0), x_2(0)) = (1, 1)\).
   (b) \((x_1, x_2)' = (x_1^2, x_2 + x_1^{-1}), \ (x_1(0), x_2(0)) = (1, 1)\).

(3) Let \(f : \mathbb{R} \to \mathbb{R}\) be continuous with \(f(0) = 0\) and \(f > 0\) on \((-\infty, 0)\) and \(f < 0\) on \((0, \infty)\).
   (a) Show that a solution of \(x' = f(x)\) must be non negative or non positive.
   (b) Give examples of \(f\) such that solutions of \(x' = f(x), \ x(0) = x_0\) with \(x_0 \neq 0\), attain the value zero.
   (c) Give examples of \(f\) such that solutions of \(x' = f(x), \ x(0) = x_0\) with \(x_0 \neq 0\), can not attain the value zero.

(4) Let \(E \subset \mathbb{R}^n\) be open and \(f : E \to \mathbb{R}^n\) be locally Lipschitz. Assume that \(x : (\alpha, \beta) \to E\) is a solution of \(x' = f(x)\) such that \(x_{\infty} := \lim_{t \to \beta} x(t)\) exists and belongs to \(E\). Show that \(\beta = \infty\) and \(f(x_{\infty}) = 0\).