Homework 3 Due September 21, 2007

(1) Sketch the phase portrait of a planar system having

(a) a trajectory $\Gamma$ with $\alpha(\Gamma) = \omega(\Gamma) = \{z\}$ but $\Gamma \neq \{z\}$.
(b) a trajectory $\Gamma$ such that $\omega(\Gamma)$ consists of one limit orbit.
(c) a trajectory $\Gamma$ such that $\omega(\Gamma)$ consists of one limit orbit and one fixed point.
(d) a trajectory $\Gamma$ such that $\omega(\Gamma)$ consists of two limit orbits and one fixed point.
(e) a trajectory $\Gamma$ such that $\omega(\Gamma)$ consists of two limit orbits and two fixed points.

(2) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be $n$-times continuously differentiable. Let $p \in \mathbb{R}$ be given with $f^{(k)}(p) = 0$ for $k = 0, \ldots, n - 1$ and $f^{(n)}(p) \neq 0$. What can you say about stability properties of the fixed point $p$ depending on $n$?

(3) Investigate the following differential equations in the plane:

(a) $x' = xy^2 - \frac{1}{2}x^3$, $y' = -\frac{1}{2}y^3 + \frac{1}{3}x^2y$.
(b) $x' = xy^4$, $y' = x^4y$.

(4) (a) Let $(M, d)$ be a compact metric space and $C : M \rightarrow M$ be a map with $d(C(x), C(y)) < d(x, y)$ for all $x, y \in M$. Show that $C$ has a unique fixed point and $C^n(x)$ converges to this fixed point for $n \rightarrow \infty$ for any $x \in M$. (Hint: Consider the function $x \mapsto d(x, C(x))$).

(b) Let $\Phi$ be a flow on $E \subset \mathbb{R}^d$ open. Let $A \subset E$ be a compact $\Phi$-invariant set with $\|\Phi_t(x) - \Phi_t(y)\| < \|x - y\|$ for all $x, y \in A$ and every $t \geq 0$. Show that $A$ contains a unique fixed point $x_F$ of $\Phi$ and that $\Phi_t(x) \rightarrow x_F$, $t \rightarrow \infty$, for any $x \in A$. 
