Math 111 Fall 2009 Final Exam Practice

Instructor: Dr. O'Donnol

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **three hours**. Do all 12 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. An answer with no supporting work or explanation will receive little to no credit.
Please print you name clearly here.

Print name: ______________________________

The solutions are on the last page but many cases do not show a full solution because the work is not shown. The actual final will have a similar number of problems but will not be as long.

1. [30 points] For the function,

$$f(x) = x^2 e^{-2x}$$

find the following:

(a) Domain:

(b) x-intercepts and y-intercepts:

(c) all asymptotes:

(d) Find the intervals where f is increasing and decreasing.

(e) (This is problem 1 cont. $f(x) = x^2e^{-2x}$)Determine the local maxima and minima using the first derivative test.

(f) Sketch the curve for the function.

2. [30 points] For the function,

$$y = \frac{x-1}{x^2}$$

find the following:

(a) Domain:

(b) x-intercepts:

(c) y-intercepts:

(d) all asymptotes:

(e) Find the intervals where y is increasing and decreasing.

(f) (This is problem 2 cont.$y = \frac{x-1}{x^2}$.) Find the concavity of y on the appropriate intervals, and inflection points.

(g) Determine the local maxima and minima using the second derivative test.

(h) Sketch the curve for the function.

3. [15 points]

Find the derivatives:

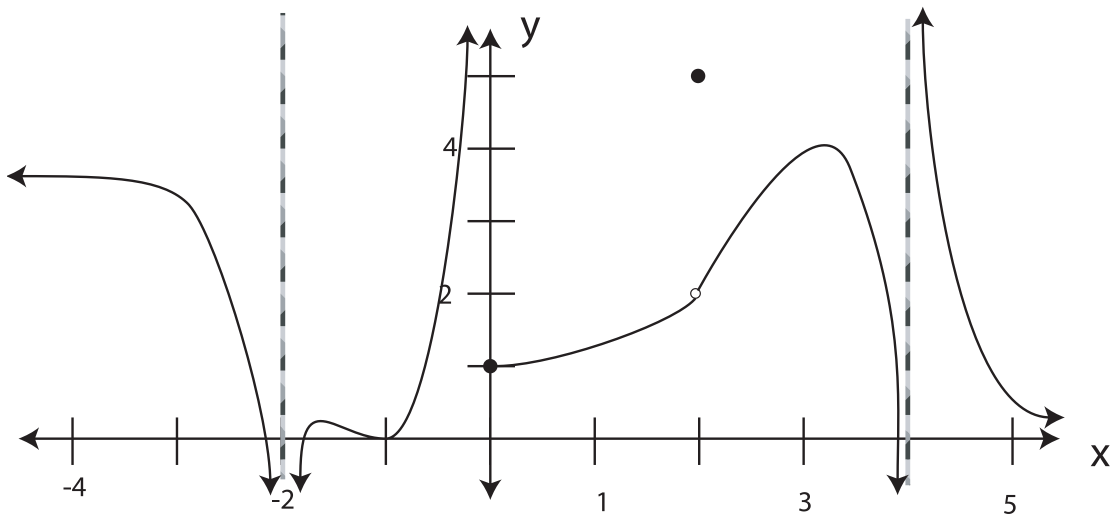
(a) $f(x) = \frac{\tan^2 x + x^7}{\ln x + 80}$

(b) $g(x) = e^{x^x}$

(c) $y = \ln\left((x^2+7)^9(x^3+1)^2\right)$

4. [15 points]

5. For the graph of the function $g(x)$ answer the following:



(a) $g(-1) =$

(b) $g'(-1) =$

(c) $g(2) =$

(d) $\lim_{x \to -2} g(x) =$

(e) $\lim_{x \to 0+} g(x) =$

(f) What are the intervals where g is continuous?

6. [10 points]

(a) Define the derivative of a function f at a point $x = a$.

(b) Use the definition of the derivative to find $f'(2)$ for $f(x) = x^3$.

7. [10 points] Find the absolute maximum and the absolute minimum values of $f(x) = x - \ln x$ on the interval $[\frac{1}{2}, 2]$. (Give that $\ln 2 = 0.693147...$)

8. [10 points] If $2 \leq f'(x) \leq 4$ for all x, show that $14 \leq f(9) - f(2) \leq 28$.

9. [15 points] Carefully, find the limits,

(a)

$$\lim_{x\to\infty} \left(e^x + x\right)^{\frac{1}{x}} =$$

(b)

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) =$$

10. [10 points] A snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm. (Surface area of a sphere is $A = 4\pi r^2$.)

11. [15 points] Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.

12. [10 points] Find the limits:

(a) $\lim_{x\to 0} \frac{5^x - 3^x}{x} =$

(b) $\lim_{x\to \pi} \sin(x + \sin x) =$

(c) $\lim_{x\to \infty} \frac{(\ln x)^2}{x} =$

Solutions:

1. [30 points]

   (a) Domain: $\mathbf{R}$

   (b) x-intercepts and y-intercepts: $(0,0)$

   (c) all asymptotes: Horizontal asymptote at $y = 0$

   (d) The function f is increasing on $(0,1)$ and decreasing on $(-\infty,0)$ and $(1,\infty)$.

   (e) f has a local min at $(0,0)$, and a local max at $(1,\frac{1}{e^2})$.

   (f) Sketch the curve for the function.

2. [30 points]

   (a) Domain: $(-\infty,0)\bigcup(0,\infty)$

   (b) x-intercepts: $(1,0)$

   (c) y-intercepts: None

   (d) all asymptotes: Vertical asymptote at $x = 0$.

   (e) The function y is increasing on $(0,2)$ and decreasing on $(-\infty,0)$ and $(2,\infty)$.

   (f) The function y is concave up on $(3,\infty)$ and concave down on $(-\infty,0)$ and $(0,3)$. It has an inflection point at $(3,\frac{2}{9})$.

   (g) It has a local maxima at $(2,\frac{1}{4})$.

   (h) Sketch the curve for the function.

3. [15 points]

   (a)
   $$f'(x) = \frac{(2\tan x\sec^2 x + 7x^6)(\ln x + 80) - (\tan^2 x + x^7)(\frac{1}{x})}{(\ln x + 80)^2}$$

   (b)
   $$g'(x) = e^{x^x} x^x(\ln x + 1)$$

   (c)
   $$y' = \frac{9(2x)}{x^2+7} + \frac{2(3x^2)}{x^3+1}$$

4. [15 points]

   (a) $g(-1) = 0$

   (b) $g'(-1) = 0$

   (c) $g(2) = 5$

   (d) $\lim_{x\to -2} g(x) = -\infty$

   (e) $\lim_{x\to 0+} g(x) = 1$

   (f) $(-\infty,-2)\bigcup(-2,0)\bigcup[0,2)\bigcup(2,4)\bigcup(4,\infty)$

5. [10 points]

   (a) The derivative of a function f at a point $x = a$ is

   $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   if the limit exists.

   (b) The derivative

   $$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{x^3 - 2^3}{x - 2} = \lim_{x \to 2} x^2 + 2x - 4 = 4$$

6. [10 points] The function $f(x) = x - \ln x$ has an absolute max value of $2 - \ln 2$ at $x = 2$ and an absolute min value of 1 at $x = 1$, on the interval $[\frac{1}{2}, 2]$.

7. [10 points] Since $2 \leq f'(x) \leq 4$ for all x, f must be both continuous and differentiable for all x. So by the MVT there is a c in $(2, 9)$ such that $f'(c) = \frac{f(9) - f(2)}{9 - 2}$. So we have

   $$7f'(c) = f(9) - f(2).$$

   Next, since $2 \leq f'(x) \leq 4$ for all x, it holds for c

   $$2 \leq f'(c) \leq 4.$$

   So

   $$14 \leq 7f'(c) \leq 28,$$

   thus

   $$14 \leq f(9) - f(2) \leq 28.$$

8. [15 points]

   (a) $\lim_{x \to \infty} \left(e^x + x\right)^{\frac{1}{x}} = e$

   (b) $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$, here you must use the Squeeze theorem.

9. [10 points] The diameter decreases at a rate of $\frac{1}{20\pi} cm/min$ when the diameter is 10 cm.

10. [15 points] First show that the equation $2x - 1 - \sin x = 0$ has at least one real root:

Let $f(x) = 2x - 1 - \sin x$. The function f is continuous as it is the sum of continuous functions,

$$f(0) = -1 < 0$$

and

$$f(\pi) = 2\pi - 1 > 0.$$

So by the IVT there is a a in $(0, \pi)$ such that $f(a) = 0$, i.e. f has at least one real root.

Now, show there cannot be more than one real root:

Suppose it has more than one real root, say a and b. Then $f(a) = f(b) = 0$, and we know that f is continuous and differentiable. So by Rolle's Theorem there is a c in (a, b) such that $f'(c) = 0$. Now $f(x) = 2x - 1 - \sin x$, so $f'(x) = 2 - \cos x$. Since we know $-1 \leq \cos x \leq 1$, then it is clear that $f'(x) > 0$. So there cannot be a c where $f'(c) = 0$, so there cannot be $f(a) = f(b) = 0$. So there cannot be more than one real root. There is exactly one real root.

11. [10 points]

    (a) $\lim_{x \to 0} \frac{5^x - 3^x}{x} = \ln 5 - \ln 3$

    (b) $\lim_{x \to \pi} \sin(x + \sin x) = 0$

    (c) $\lim_{x \to \infty} \frac{(\ln x)^2}{x} = 0$