Rice University	Math 111 - Fall 2008
Department of Mathematics	Final Exam
Dr. Elena Pavelescu	Solutions

This exam is 2 hours 30 minutes long. No notes, books or calculators are allowed. You may use your handwritten index card.

1. *(8 points)* Find the domain and range of the following functions:

$$f(x) = \sqrt{x^2 - 1}$$

Domain (f)=$\{x \text{ with } x^2 - 1 \geq 0\}$=$\{x \text{ with } x^2 \geq 1\}$ =$(-\infty, -1] \cup [1, \infty)$.
Range (f)=$[0, \infty)$

$$g(x) = \ln(2x + 1)$$

Domain (g)=$\{x \text{ with } 2x + 1 > 0\}$=$(-\frac{1}{2}, \infty)$.
Range(g)=$\mathbb{R}$

2. *(5 points)* Use the intermediate value property to show that the equation

$$x = \cos x$$

has at least one solution in the interval $[0, \frac{\pi}{2}]$.

Solution: Consider the function $f(x) = x - \cos x$ defined on the interval $[0, \frac{\pi}{2}]$. This is a continuous function and thus it satisfies the Intermediate Value Property.
$f(0) = 0 - \cos 0 = -1 < 0$
$f(\frac{\pi}{2}) = \frac{\pi}{2} - \cos \frac{\pi}{2} = \frac{\pi}{2} - 0 = \frac{\pi}{2} > 0$.
As $f(0) < 0$ and $f(\frac{\pi}{2}) > 0$, IVP implies there exists a $c \in [0, \frac{\pi}{2}]$ such that $f(c) = 0$. This is equivalent to $c - \cos c = 0$, or $c = \cos c$. The values c is the solution we are looking for.

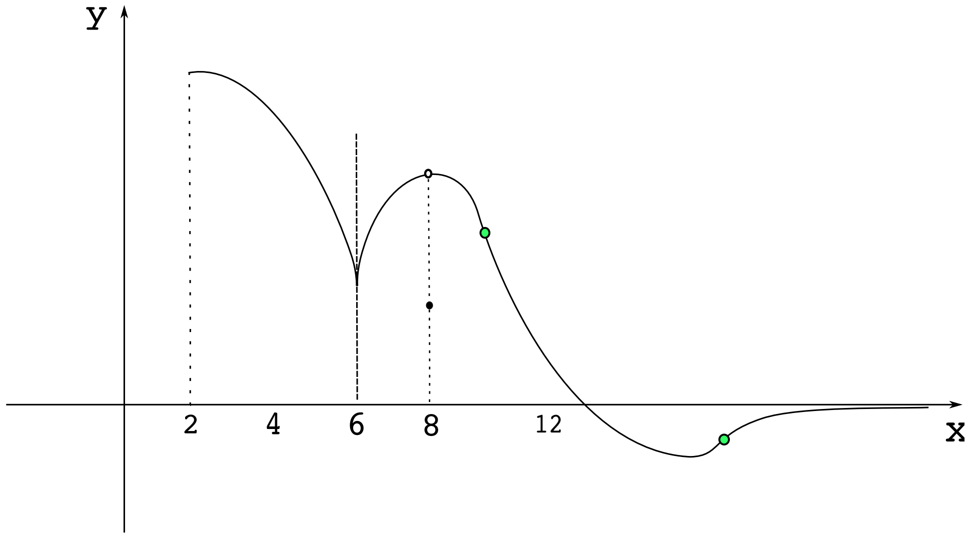
3. *(14 points)* Below you are given the graph of the function $f(x)$ defined on the interval $[2, \infty)$. Decide if the following statements are true or false. If a statement is true, just write TRUE next to it. If a statement is false, write FALSE and briefly EXPLAIN why you think it is false.

A) f has a limit at $x = 8$
TRUE. The function f approaches the same y-values from the left and right of $x = 8$ i.e

$$\lim_{x \to 8^-} f(x) = \lim_{x \to 8^+} f(x)$$

B) f is continuous at $x = 8$
FALSE.

$$f(8) \neq \lim_{x \to 8^-} f(x) = \lim_{x \to 8^+} f(x)$$



C) f is differentiable at $x = 6$
FALSE. The graph of f has a vertical tangent at $x = 6$ and so it doesn't have a well defined slope here.

D) $f'(12) > 0$
FALSE. The f is decreasing in a small neighborhood of $x = 12$, thus $f'(12) < 0$.

E) $f''(4) < 0$
TRUE. The graph of f is concave down for a small neighborhood of $x = 4$, thus $f''(4) < 0$.

F) The graph of f has 3 inflection points
FALSE. The graph of f has 2 inflection points, as marked with green dots.

G)

$$\lim_{x\to\infty} f(x) = 0$$

TRUE. The line $y = 0$ is a horizontal asymptote for the graph of $f(x)$ as $x \to \infty$.

4. *(10 points)* Find the absolute minimum and the absolute maximum of

$$g(x) = \ln x + \frac{2}{x}$$

on the interval $[1, e^2]$.

Solution:

The function $g(x) = \ln x + \frac{2}{x}$ is continuous on the interval $[1, e^2]$, so the absolute minimum and the absolute maximum happen either at a critical point or at the endpoints.

$g'(x) = \frac{1}{x} - \frac{2}{x^2} = \frac{x-2}{x^2}$. The critical point is $x = 2 \in [1, e^2]$. The function is not defined at $x = 0$ so we ignore this value and we check $x = 1, 2, e^2$.

$g(1) = \ln 1 + \frac{2}{x} = 0 + 2 = 2$
$g(2) = \ln 2 + \frac{2}{2} = \ln 2 + 1$
$g(e^2) = \ln e^2 + \frac{e^2}{2} = 2 + \frac{e^2}{2}$

As $2 < e$, $\ln 2 < \ln e$ and so $\ln 2 + 1 < \ln e + 1 = 2$.

We have $\ln 2 + 1 < 2 < 2 + \frac{e^2}{2}$.

Global minimum : $g(2) = \ln 2 + 1$. Global maximum : $g(e^2) = 2 + \frac{e^2}{2}$.

5. *(12 points)* Compute the following derivatives:

A) $f(x) = \frac{x^3}{x+1}$, $f'(x) = \frac{3x^2(x+1) - x^3 \cdot 1}{(x+1)^2} = \frac{2x^3+3x^2}{(x+1)^2}$ (we used the quotient rule)

B) $g(x) = \sin 3x \cdot e^{2x}$, $g'(x) = \cos 3x \cdot 3 \cdot e^{2x} + \sin 3x \cdot e^{2x} \cdot 2$ (we used the product and chain rules)

C) $h(x) = \sqrt{\ln 3x}$, $h'(x) = \frac{1}{2}(\ln 3x)^{-1/2} \cdot \frac{1}{3x} \cdot 3 = \frac{1}{2x\sqrt{\ln 3x}}$ (we used the chain rule twice)

D) $i(x) = x^{2x}$ (use logarithmic differentiation). We take ln of both sides and get

$$\ln(i(x)) = \ln(x^{2x}) = 2x \ln x$$

We take the derivative with respect to x here and get

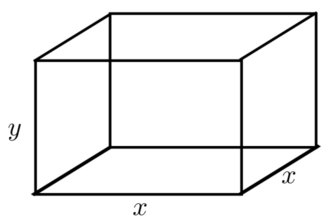
$$\frac{1}{i(x)} \cdot i'(x) = 2\ln x + 2x\frac{1}{x} = 2\ln x + 2$$

We solve for $i'(x)$ and get $i'(x) = i(x) \cdot (2\ln x + 2) = x^{2x}(2\ln x + 2)$

6. *(10 points)* A rectangular block with square base of length x is being squeezed in such a way that its height y is decreasing at the rate of 2 cm/min while its volume remains constant.

A) *(2 points)* Draw the box, label the edges, and write the expression of the volume V in terms of x and y.

B) *(2 points)* As V remains constant, $\frac{dV}{dt} = 0$



$V = x^2 y$

C) *(6 points)* At what rate is the edge x changing when $x = 30$ cm and $y = 20$ cm?

We take the derivative with respect to the time variable t in the expression of the volume and we get

$$\frac{dV}{dt} = 2x\frac{dx}{dt}y + x^2\frac{dy}{dt}$$

We substitute $\frac{dV}{dt} = 0$, $x = 30$, $y = 20$, $\frac{dy}{dt} = -2$ in the above equation and get

$$0 = 2 \cdot 30 \cdot \frac{dx}{dt} \cdot 20 - 30^2 \cdot 2$$

We solve for $\frac{dx}{dt}$ and get $\frac{dx}{dt} = \frac{3}{2}$ cm/min. At the specified time, the x edge increases at a rate of $\frac{3}{2}$ cm/min.

7. *(5 points)* Use the linear approximation of an appropriate function to approximate $\sqrt[3]{28}$.

Solution:

We use the linear approximation of $f(x) = \sqrt[3]{x}$ at the point $x = 27$ and we get

$$\sqrt[3]{x} = f(x) \sim f(27) + f'(27)(x - 27)$$

We have $f'(x) = \frac{1}{3}x^{-2/3}$ and so $f'(27) = \frac{1}{3}27^{-2/3} = \frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}$. For $x = 28$ we get

$$\sqrt[3]{28} = f(28) \sim f(27) + f'(27))(28 - 27) = \sqrt[3]{27} + \frac{1}{27} = 3 + \frac{1}{27} = \frac{82}{27}$$

8. *(12 points)*The height $y(t)$ (in feet at time t seconds) of a ball thrown vertically upward is given by $y(t) = -16t^2 + 96t + 50$.

A) *(4 points)* Find the maximum height that the ball attains.

We look for the critical points of $y(t)$: $y'(t) = -32t + 96 = -32(t - 3) = 0$ implies $t = 3$.
The graph is a parabola pointing downward so we know this is the global maximum. Or, the second derivative of y is $y''(t) = -32 < 0$, thus the graph is concave down all the way and so $y(3) = -16 \cdot 9 + 96 \cdot 3 + 50 = 194$ is a global maximum for the height function.

B) *(4 points)* Use the Mean Value Theorem to show that in the time interval from $t = 1$ to $t = 2$ there is at least a time t_0 when the velocity of the ball is exactly 48ft/sec.

The Mean Value Theorem applied to $y(t)$ says that there exists a time $t_0 \in [1, 2]$ such that

$$y'(t_0) = \frac{y(2) - y(1)}{2 - 1} = \frac{-16 \cdot 4 + 96 \cdot 2 - (-16 + 96)}{1} = 48ft/sec$$

C) *(4 points)* Find the time t_0 from B).

$$y'(t_0) = -32t_0 + 96 = 48$$

We solve for t_0 and find $t_0 = \frac{3}{2}$ sec. At this exact time the velocity is 48 ft/sec.

9. *(12 points)* Compute the following limits or show they do not exist:

A)

$$\lim_{x \to \sqrt{5}} \frac{x^2 - 5}{x - \sqrt{5}} = \lim_{x \to \sqrt{5}} \frac{(\sqrt{x} - \sqrt{5})\sqrt{x} + \sqrt{5})}{x - \sqrt{5}} = \lim_{x \to \sqrt{5}} (x + \sqrt{5}) = 2\sqrt{5}$$

B)

$$\lim_{x \to \pi} \frac{x}{\sin x} DNE$$

$$\lim_{x \to \pi^-} \frac{x}{\sin x} = +\infty$$

while

$$\lim_{x \to \pi^+} \frac{x}{\sin x} = -\infty$$

This is because $\sin x$ is positive on the left of π and negative on the right of π.

C)

$$\lim_{x \to \infty} \frac{e^{2x}}{x^2 + 1} \overset{\frac{\infty}{\infty}}{=}_{LH}= \lim_{x \to \infty} \frac{e^{2x} \cdot 2}{2x} \overset{\frac{\infty}{\infty}}{=}_{LH}= \lim_{x \to \infty} \frac{e^{2x} \cdot 2 \cdot 2}{2} = \lim_{x \to \infty} 2e^{2x} = \infty$$

D)

$$\lim_{x\to 0^+}(\sin x \cdot \ln x) = 0\cdot\infty = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sin x}} \overset{\frac{\infty}{\infty}}{=}_{LH}= \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}} = \lim_{x\to 0^+} -\frac{\sin^2 x}{x\cos x} \overset{\frac{0}{0}}{=}_{LH}=$$

$$= \lim_{x\to 0^+} -\frac{2\sin x\cos x}{\cos x - x\sin x} = \lim_{x\to 0^+}\frac{0}{1-0} = 0$$

10. *(12 points)* Consider $f(x) = 3x^5 - 20x^3$, $x \in \mathbb{R}$.

A) Find the critical points of f and classify them as local minimum, local maximum or neither
B) Find the intervals where f is increasing and those where f is decreasing
C) Find the inflection points of f
D) What happens with the graph of f as $x \to \infty$ and as $x \to -\infty$?
E) Sketch the graph of f.

Solution:

A) $f'(x) = 15x^4 - 60x^2 = 15x^2(x^2-4) = 15x^2(x-2)(x+2)$. Critical points : $x = 0, 2, -2$.

B) $f(-2) = 64$ is a local maximum

x	x^2	$x-2$	$x+2$	$f'(x)$	$f(x)$
$(-\infty,-2)$	+	−	−	+	increasing
$(-2,0)$	+	−	+	−	decreasing
$(0,2)$	+	−	+	−	decreasing
$(2,\infty)$	+	+	+	+	increasing

$f(0) = 0$ is not a local extremum
$f(2) = -64$ is a local minimum

C)$f''(x) = 60x^3 - 120x = 60x(x^2-2) = 60x(x-\sqrt{2})(x+\sqrt{2}) = 0$ implies $x = 0, \sqrt{2}, -\sqrt{2}$ are the possible inflection points. We check the sign of f'' to see if the graph of f changes concavity at this points.

x	x	$x-\sqrt{2}$	$x+\sqrt{2}$	$f''(x)$	concavity
$(-\infty,-\sqrt{2})$	−	−	−	−	$\cap$
$(-\sqrt{2},0)$	−	−	+	+	$\cup$
$(0,\sqrt{2})$	+	−	+	−	$\cap$
$(\sqrt{2},\infty)$	+	+	+	+	$\cup$

$f(0) = 0$
$f(\sqrt{2}) = -28\sqrt{2}$
$f(-\sqrt{2}) = 28\sqrt{2}$ We have inflection points at $(0, f(0))$, $(\sqrt{2}, f(\sqrt{2}))$ and $(-\sqrt{2}, f(-\sqrt{2}))$
D)

$$\lim_{x\to+\infty} f(x) = +\infty$$

$$\lim_{x\to-\infty} f(x) = -\infty$$

E)