Rice University	Math 111 - Fall 2008
Department of Mathematics	Midterm Exam 1
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1.Decide if the following statements are true or false. If a statement is true, just write TRUE next to it. If a statement is false, write FALSE and explain briefly why you think it is false.

A) The graph of the equation $(x-4)^2 + (y+3)^2 = 4$ is a circle or radius 2 centered at $(-4, 3)$.
FALSE. The graph of this equation is indeed a circle of radius 2, but centered at (4,-3).

B) The function $f(x) = \frac{\cos x}{\sin x}$ is defined for all real values of x.
FALSE. The function f is only defined for those $x \in \mathbb{R}$, for which $\sin x \neq 0$. That is all $x \in \mathbb{R}$, $x \neq k\pi$, for $k \in \mathbb{Z}$.

C) If a function f is differentiable at a point a then f is also continuous at a.
TRUE, by definition we can only talk about differentiability at those points where f is continuous.

2. Let $f(x) = \sin(2x)$ and $g(x) = \frac{1}{x^2+1}$.

A) What are the domain and range of f and g?

Domain(f)=$\mathbb{R}$, Range(f)=[-1,1].

Domain(g)=$\mathbb{R}$, Range(g) $= \{0 < x \leq 1\}$.

B) Draw the graph of f.

C) Compute $f \circ g$ and $g \circ f$.

$(f \circ g)(x) = f(g(x)) = f(\frac{1}{x^2+1}) = \sin(\frac{2}{x^2+1})$

$(g \circ f)(x) = g(f(x)) = g(\sin(2x)) = \frac{1}{\sin^2(2x)+1}$.

3. Compute the following limits or show that they do not exist.

A)

$$\lim_{x\to 2} \frac{x^2 - 4x + 4}{x - 2} = \lim_{x\to 2} \frac{(x-2)^2}{x-2} = \lim_{x\to 2}(x-2) = 2 - 2 = 0$$

B)

$$\lim_{x\to -1} \frac{1}{x+1}$$

limit doesn't exist because

$$\lim_{x\to -1^-} \frac{1}{x+1} = -\infty$$
$$\lim_{x\to -1^+} \frac{1}{x+1} = +\infty$$

C) We use the limits

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1, \lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1$$

$$\lim_{x\to 0} \frac{\tan(2x)}{\sin(4x)} = \lim_{x\to 0} \frac{\tan(2x)}{2x} \frac{4x}{\sin(4x)} \frac{2x}{4x} = 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$$

Note that the first two quotients limit to 1 and the third quotient simplifies to $\frac{1}{2}$.

4. You are given the graph a function. Label each of the x values given below with the appropriate statement. Choose from:

A. f doesn't have a limit at this point
B. f has a limit but is not continuous at this point
C. f is continuous at this point

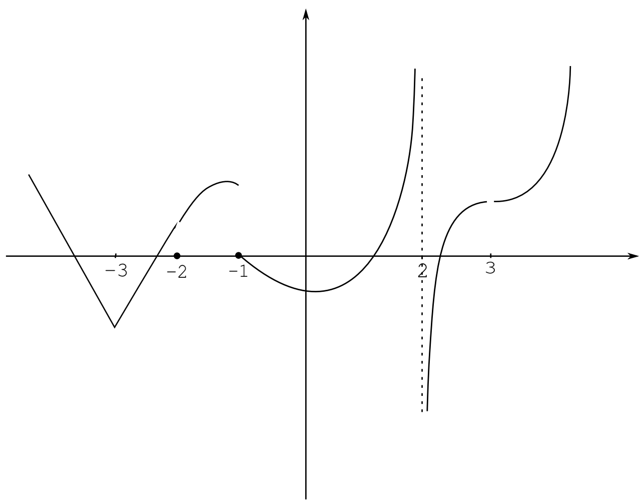

Figure 1: Graph of f

$x = -3$: (C)f is continuous at this point
$x = -2$: (B)f has a limit but is not continuous at this point
$x = -1$: (A)f doesn't have a limit at this point
$x = 2$: (A)f doesn't have a limit at this point
$x = 3$: (B)f has a limit but is not continuous at this point

5. Use the intermediate value property of continuous functions to show that the equation $\frac{x}{6} + \cos x = 0$ has at least 2 distinct solutions in the interval $[0, 2\pi]$. (Hint: It might help to evaluate the right hand side at $x = \pi$).

Let $f(x) = \frac{x}{6} + \cos x$.

$f(0) = 0 + \cos(0) = 0 + 1 = 1 > 0$
$f(\pi) = \frac{\pi}{6} + \cos\pi = \frac{\pi}{6} + (-1) = \frac{\pi}{6} - 1 < 0$
$f(2\pi) = \frac{2\pi}{6} + \cos(2\pi) = \frac{\pi}{3} + 0 > 0.$

As f is continuous on $[0, \pi]$, by IVP, for $K = 0$ which is between $f(0) = 1$ and $f(\pi) = \frac{\pi}{6} - 1$, there exists some $c_1 \in [0, \pi]$ for which $f(c_1) = 0$. This is the first solution.
As f is continuous on $[\pi, 2\pi]$, by IVP, for $K = 0$ which is between $f(\pi) = \frac{\pi}{6} - 1$ and $f(2\pi) = \frac{\pi}{3}$, there exists some $c_2 \in [\pi, 2\pi]$ for which $f(c_2) = 0$. This is the second solution.
The two values, c_1 and c_2 are distinct as neither is equal to π and the two intervals $[0, \pi]$ and $[\pi, 2\pi]$ are otherwise disjoint.

6. Suppose a sprinter running a 100 -meter race is $S(t) = \frac{1}{4}(t^2 + 32t)$ meters away from the starting line at t seconds after the start of the race.

A) What is the sprinter's average velocity during the first 4 seconds of the race (AV)?

$$AV = \frac{S(4) - S(0)}{4 - 0} = \frac{\frac{1}{4}(4^2 + 32 \cdot 4) - 0}{4} = \frac{36}{4} = 9m/s$$

B) Use the definition of the derivative to find the sprinter's velocity at time $t = 8$.

The sprinter's velocity at $t = 8$ is:

$$S'(8) = \lim_{h \to 0} \frac{S(8 + h) - S(8)}{h} = \lim_{h \to 0} \frac{\frac{1}{4}((8 + h)^2 + 32(8 + h)) - \frac{1}{4}(64 + 32 \cdot 8)}{h} =$$

$$= \lim_{h \to 0} \frac{\frac{1}{4}(64 + 16h + h^2 + 32 \cdot 8 + 32h - 64 - 32 \cdot 8)}{h} = \lim_{h \to 0} \frac{\frac{1}{4}(48h + h^2)}{h} = \lim_{h \to 0} \frac{1}{4}(48 + h) = 12m/s$$