COMPOSITION VERSUS PRODUCT OF FUNCTIONS

It's important to realize that given two functions, f and g one may consider their product and their composition. These are two distinct operations and are defined as follows.

THE PRODUCT is given by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

where the $\cdot$ is the usual multiplication.

The domain of the product function $f \cdot g$ is composed of all those values of x that are both in Domain(f) and in Domain(g).

On the other hand, THE COMPOSITION is given by

$$(f \circ g)(x) = f(g(x))$$

The domain of the composition is composed of all those values of x that are in Domain(g) such that $g(x)$ is in Domain(f).

EXAMPLE

Say $f(x) = \frac{1}{x+1}$ and $g(x) = \frac{1}{x-2}$.

Domain(f): $\mathbb{R}$, $x \neq -1$ and Domain(g): $\mathbb{R}$, $x \neq 2$.

The product of f and g is:

$$(f \cdot g)(x) = f(x) \cdot g(x) = \frac{1}{x+1} \cdot \frac{1}{x-2}$$

Domain($f \cdot g$) is composed of all those values of x that are both in Domain(f) and in Domain(g), that is Domain($f \cdot g$): $\mathbb{R}$, $x \neq -1$, $x \neq 2$.

The composition is:

$$(f \circ g)(x) = f(g(x)) = f(\frac{1}{x-2}) = \frac{1}{\frac{1}{x-2} + 1}.$$

Domain($f \circ g$): Domain(g), except those x that make $\frac{1}{x-2} + 1 = 0$ (this is the same as $g(x) = -1$ which is not in Domain(f)).

$\frac{1}{x-2} + 1 = \frac{1}{x-2} + \frac{x-2}{x-2} = \frac{1+x-2}{x-2} = \frac{x-1}{x-2}$. This becomes 0 if $x = 1$.

Thus, Domain($f \circ g$): $\mathbb{R}$, $x \neq 2$, $x \neq 1$. In other words, we need to restrict Domain(g) for $f \circ g$ to make sense.

We can further simplify the the expression of $(f \circ g)(x)$ to

$(f \circ g)(x) = \frac{1}{\frac{1}{x-2}+1} = \frac{1}{\frac{x-1}{x-2}} = \frac{x-2}{x-1}$.