

First Midterm Exam: Solutions
Math 212 Fall 2010

Instructions: This is a **90 minute** exam. You should work alone, without access to any book or notes. No calculators are allowed. Do not discuss this exam with anyone other than your instructor. When you have completed the exam, write out and sign the Honor Code pledge on the front.

The exam consists of 6 questions. You must show all of your work on each problem to receive full credit, and **be sure to clearly indicate your final answer** to each question.

1. [15 Points]

(a) Compute the dot product $(3\mathbf{i} - 2\mathbf{k}) \cdot (\mathbf{i} - 6\mathbf{j} + 3\mathbf{k})$.

Solution: $3 \cdot 1 + 0 \cdot (-6) + (-2) \cdot 3 = \boxed{-3}$.

(b) Compute the determinant $\begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & 0 \\ 3 & 1 & -2 \end{vmatrix}$.

Solution: We expand along the second row according to the signs $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$:

$$\begin{aligned} \begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & 0 \\ 3 & 1 & -2 \end{vmatrix} &= -2 \begin{vmatrix} -1 & 3 \\ 1 & -2 \end{vmatrix} + 0 \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} - 0 \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} \\ &= (-2)((-1)(-2) - 3 \cdot 1) \\ &= \boxed{2}. \end{aligned}$$

(c) Let $\mathbf{u} = (1, -1, 2)$, $\mathbf{v} = (1, -3, 1)$, and $\mathbf{w} = (2, -2, 4)$. Compute $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Solution: The quantity $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is equal to the determinant of the 3×3 matrix whose rows are the vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} . Since one row is a scalar multiple of another, the determinant is $\boxed{0}$.

2. [15 Points]

(a) Let $f(x, y, z) = x \sin yz + z^3 \ln y$. Compute the gradient ∇f .

Solution:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \sin yz \\ \frac{\partial f}{\partial y} &= xz \cos yz + \frac{z^3}{y} \\ \frac{\partial f}{\partial z} &= xy \cos yz + 3z^2 \ln y, \text{ so} \\ \nabla f &= \left(\sin yz, \quad xz \cos yz + \frac{z^3}{y}, \quad xy \cos yz + 3z^2 \ln y \right).\end{aligned}$$

(b) Let $\mathbf{g}(x, y, z) = (xy - e^{yz}, z \sin xy^2)$. Compute the matrix of partial derivatives \mathbf{Dg} .

Solution:

$$\begin{aligned}\mathbf{Dg} &= \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} y & x - ze^{yz} & -ye^{yz} \\ y^2 z \cos xy^2 & 2xyz \cos xy^2 & \sin xy^2 \end{bmatrix}\end{aligned}$$

3. [20 Points] Suppose that the temperature at a point (x, y) , $x > 0$ in the plane is given by $T = \sqrt{x} + e^{x^2 y}$ and that an object is moving in the plane according to the path $\mathbf{c}(t) = (1 - \cos 3t, \cos t)$. Let $T(t)$ be the temperature the object experiences at time t .

(a) Using the multivariable chain rule, find the rate of change $\frac{dT}{dt}$ of the temperature experienced by the object at time $t = \pi/2$.

Solution: We compute $\nabla T = \left(\frac{1}{2\sqrt{x}} + 2xye^{x^2 y}, x^2 e^{x^2 y} \right)$, so that

$$\begin{aligned}\nabla T(\mathbf{c}(t)) &= \left(\frac{1}{2\sqrt{1 - \cos 3t}} + 2(1 - \cos 3t)(\cos^2 t)e^{(1 - \cos 3t)^2 \cos t}, (1 - \cos 3t)^2 e^{(1 - \cos 3t)^2 \cos t} \right) \Big|_{t=\pi/2} \\ &= \left(\frac{1}{2}, 1 \right).\end{aligned}$$

We find also, $\mathbf{c}'(t) = (3 \sin 3t, -\sin t)$, so $\mathbf{c}'(\pi/2) = (-3, -1)$, and by the chain rule

$$\frac{dT}{dt}(\pi/2) = \left(\frac{1}{2}, 1 \right) \cdot (-3, -1) = \frac{1}{2}(-3) + (1)(-1) = \boxed{-\frac{5}{2}}.$$

(b) Find an approximate value for the temperature at time $t = (\pi/2) + 0.06$.

Solution:

$$\begin{aligned} T(\pi/2 + 0.06) &\approx T(\pi/2) + T'(\pi/2) ((\pi/2 + 0.06) - \pi/2) \\ &= 2 + \left(-\frac{5}{2}\right) (0.06) \\ &= \boxed{1.85} \end{aligned}$$

4. [15 Points] Compute the following multivariable limits, or show that they do not exist:

(a) $\lim_{(x,y) \rightarrow (0,0)} \frac{3x}{x^2 + y^2}$

Solution: Taking the limit along the x -axis, we find $\lim_{x \rightarrow 0} \frac{3x}{x^2} = \lim_{x \rightarrow 0} \frac{3}{x}$, which does not exist, so the multivariable limit certainly does not exist.

(b) $\lim_{(x,y) \rightarrow (0,\pi/3)} \frac{\cos(xy^2) - 1}{xy^2}$

Solution: Set $u = xy^2$. We have $\lim_{u \rightarrow 0} \frac{\cos u - 1}{u} = \lim_{u \rightarrow 0} \frac{-\sin u}{1} = 0$. Thus the function

$$f(u) = \begin{cases} \frac{\cos u - 1}{u} & u \neq 0 \\ 0 & u = 0 \end{cases}$$

is continuous for all $u \in \mathbb{R}$, and setting $g(x, y) = xy^2$, we see the composition $h = f \circ g$, defined by

$$h(x, y) = \begin{cases} \frac{\cos(xy^2) - 1}{xy^2} & x \neq 0 \text{ and } y \neq 0 \\ 0 & x = 0 \text{ or } y = 0 \end{cases}$$

is continuous for all $(x, y) \in \mathbb{R}^2$. Thus

$$\lim_{(x,y) \rightarrow (0,\pi/3)} \frac{\cos(xy^2) - 1}{xy^2} = \lim_{(x,y) \rightarrow (0,\pi/3)} h(x, y) = h(0, \pi/3) = \boxed{0}.$$

(c) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^3}{x^4 + y^6}$

Solution: Although the limit is zero if we approach $(0, 0)$ along any line, if we approach on along the curve $\mathbf{c}(t) = (t^3, t^2)$, we find

$$\lim_{t \rightarrow 0} \frac{(t^3)^2 (t^2)^3}{(t^3)^4 + (t^2)^6} = \lim_{t \rightarrow 0} \frac{t^{12}}{t^{12} + t^{12}} = \frac{1}{2}.$$

Thus the limit does not exist.

5. [15 Points]

- (a) Find the equation for a plane that is perpendicular to both the plane $x+3y-2z+4=0$ and the plane $x+2z=0$, or show that no such plane exists.

Solution: Two planes are perpendicular if and only if their normal vectors are perpendicular, so we simply need to find a vector perpendicular to both $(1, 3, -2)$ and $(1, 0, 2)$. To do this, we take the cross product $(1, 3, -2) \times (1, 0, 2) = (6, -4, -3)$, so we may take the plane $\boxed{6x - 4y - 3z + D = 0}$ for any D .

- (b) Find the equation for a plane that is perpendicular to both the plane defined by $2x + y - 3z + 1 = 0$ and the line $\mathbf{l}(t) = (1, -2, -7) + t(2, 5, 3)$, or show that no such plane exists.

Solution: If such a plane exists, the fact that it is perpendicular to the given line tells us that its normal vector must be $(2, 5, 3)$. We need only check that this normal vector is perpendicular to $(2, 1, -3)$, which we do by checking whether the dot product is zero:

$$(2, 5, 3) \cdot (2, 1, -3) = (2)(2) + (5)(1) + (3)(-3) = 0.$$

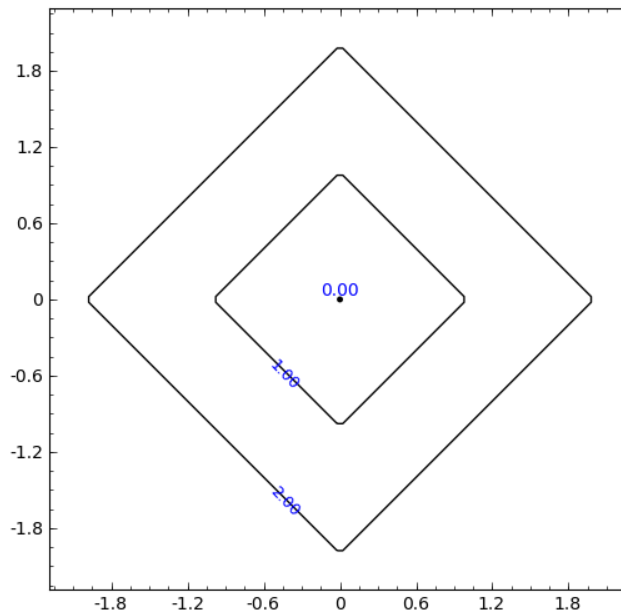
Thus we can take the plane $\boxed{2x + 5y + 3z + D = 0}$ for any D .

6. [20 Points] In this problem, we consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = |x| + |y|.$$

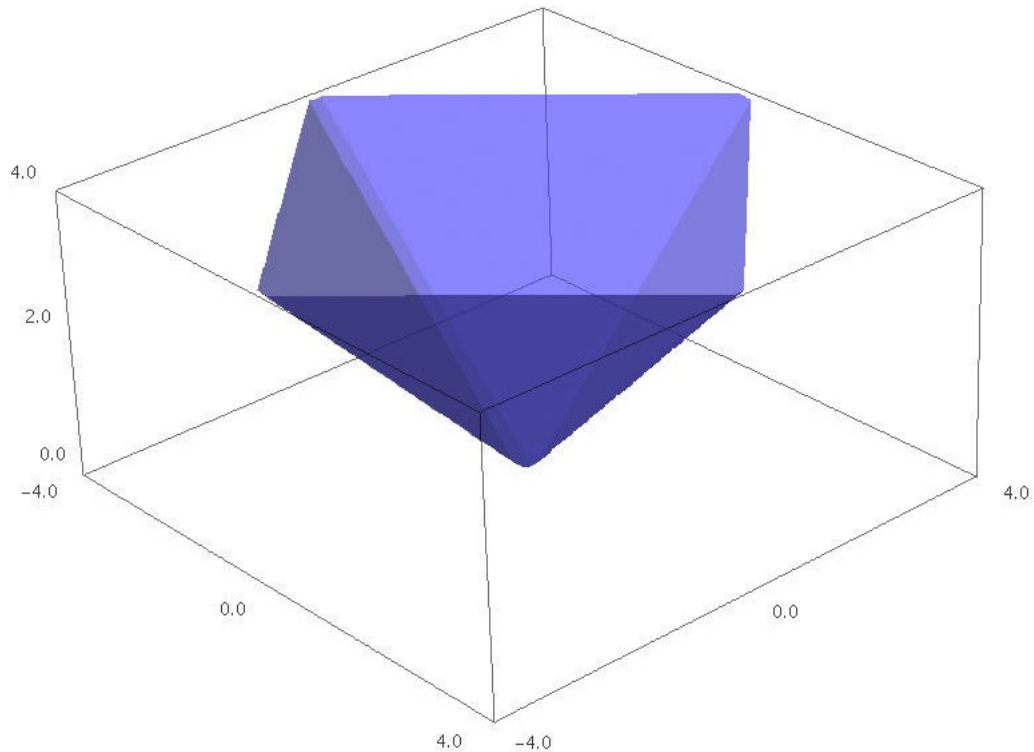
- (a) Plot the level sets $f(x, y) = c$ for $c = -1, 0, 1, 2$.

Solution: The $c = -1$ level set is empty, as the function is always nonnegative.



(b) Sketch the graph of f .

Solution:



(c) Find an equation for the tangent plane to the graph of f at the point $(1, -2, 3)$.

Solution: For (x, y) near $(1, -2)$, we have $f(x, y) = |x| + |y| = x - y$, so the graph of f near $(1, -2, 3)$ is actually a piece of the plane $z = x - y$; the tangent plane to a plane is of course¹ just that same plane, $x - y - z = 0$.

¹This agrees with our usual formula for the tangent plane involving partial derivatives since $\nabla f(1, -2) = (1, -1)$.