

MATH 366: Möbius transformations review

A *Möbius transformation* is a function T from $\hat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ to itself, defined by four complex numbers a, b, c, d with $ad - bc \neq 0$ as follows:

$$T(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq \infty \text{ and } cz+d \neq 0 \\ a/c & \text{if } z = \infty \\ \infty & \text{if } cz+d = 0, \end{cases}$$

where if $c = 0$ we say $T(\infty) = \infty$. Two Möbius transformations $\frac{az+b}{cz+d}$ and $\frac{a'z+b'}{c'z+d'}$ are equal if and only if the coefficients differ by a scalar multiple, i.e. $(a, b, c, d) = (\lambda a', \lambda b', \lambda c', \lambda d')$ for some complex number $\lambda \neq 0$.

The composition of two Möbius transformations is a Möbius transformation. Every Möbius transformation $T(z) = \frac{az+b}{cz+d}$ is a one-to-one correspondence from $\hat{\mathbf{C}}$ to itself and its inverse is the Möbius transformation

$$T^{-1}(z) = \frac{dz - b}{-cz + a}.$$

Every Möbius transformation can be written as a composition of Möbius transformations in the special forms $S_a(z) = az$, $T_b(z) = z + b$, and $U(z) = 1/z$. We can use this to simplify the proof of the following two theorems:

Theorem. *The cross-ratio is preserved by Möbius transformations. More precisely, if $z_1, z_2, z_3, z_4 \in \hat{\mathbf{C}}$ are distinct, and T is any Möbius transformation, then*

$$(z_1, z_2; z_3, z_4) = (T(z_1), T(z_2); T(z_3), T(z_4)),$$

where the cross-ratio¹ $(z_1, z_2; z_3, z_4)$ of z_1, z_2, z_3, z_4 is defined to be

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

with the usual conventions in the case where some $z_k = \infty$.

Theorem. *Any Möbius transformation induces a one-to-one correspondence from the set*

$$C = \{\text{circles in } \hat{\mathbf{C}}\} = \{\text{lines and circles in } \mathbf{C}\}^2$$

to itself. Moreover, a Möbius transformation preserves angles: if T is a Möbius transformation and C_1, C_2 are circles in $\hat{\mathbf{C}}$ that meet, then $T(C_1)$ and $T(C_2)$ meet in the same angle as C_1 and C_2 ³.

¹This is related to the *real cross-ratio* defined on page 319 of the book as follows: if A, B, P, Q are points in the plane (thought of as the complex plane \mathbf{C}), then

$$(AB, PQ) = \frac{(\overline{AP})(\overline{BQ})}{(\overline{BP})(\overline{AQ})} = \left| \frac{(A-P)(B-Q)}{(P-B)(Q-A)} \right| = |(A, P; B, Q)|,$$

but in the case where the cross-ratio is used to compute hyperbolic distance, the absolute value is unnecessary because the complex cross-ratio will already be a positive real number.

²We think of a line as a circle through ∞ .

³In order to make this true in the case of lines, we think of parallel lines as being circles tangent to one another at ∞ .

In particular, a Möbius transformation sends orthogonal circles (or lines) to orthogonal circles (or lines), and hence any Möbius transformation T that preserves the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ sends the lines of the Poincaré disk model to other P-lines. Moreover, if $A, B \in \mathbf{D}$, the hyperbolic distance from A to B is defined to be

$$d(A, B) = |\log(A, P; B, Q)|$$

where P, Q are the the points where the P-line through A and B meets the unit circle. Since T sends the P-line through A and B to the P-line through $T(A)$ and $T(B)$, and since any Möbius transformation preserves cross-ratios, T preserves hyperbolic distance, i.e.

$$d(A, B) = d(T(A), T(B)).$$

Thus we see that **Möbius transformations preserving the unit disk are motions of the Poincaré disk model of the hyperbolic plane**. In fact, the Möbius transformations T with $T(\mathbf{D}) = \mathbf{D}$ are all the *orientation-preserving* motions of the hyperbolic plane (i.e. all the motions but the reflections and glide reflections).

Using the cross-ratio for existence and the fact that a non-identity Möbius transformation has at most two fixed points for uniqueness, we can prove:

Theorem. *Let $z_1, z_2, z_3 \in \hat{\mathbf{C}}$ be distinct and $w_1, w_2, w_3 \in \hat{\mathbf{C}}$ be distinct. Then there is a unique Möbius transformation T such that $T(z_1) = w_1$, $T(z_2) = w_2$, and $T(z_3) = w_3$.*

We can use this theorem together with the following proposition to show that given any two circles C_1 and C_2 , there exists a Möbius transformation T for which $T(C_1) = C_2$:

Proposition. *For any three distinct points in $\hat{\mathbf{C}}$ there is a unique circle in $\hat{\mathbf{C}}$ (i.e. line or circle in \mathbf{C}) passing through them.*

In particular, if we set $T(\infty) = 0$, $T(0) = -1$, $T(1) = -i$, we get that

$$T(z) = \frac{z - i}{z + i}, \text{ whose inverse is } T^{-1}(z) = \frac{iz + i}{-z + 1},$$

sends the real axis to the unit circle; we can check that $T(-1) = i$, $T(i) = 0$, and T sends the upper half-plane $\mathbf{H} = \{z \in \mathbf{C} : \text{Im } z > 0\}$ to \mathbf{D} , and sends the positive imaginary axis to the interval $(-1, 1)$ of the real axis in the disk. In fact, **this transformation T induces a hyperbolic-distance-preserving isomorphism from the Poincaré upper half-plane model \mathbf{H} to the Poincaré disk model \mathbf{D} .**

While in general it is a difficult computation to write down the hyperbolic rotations about a specified point, the hyperbolic translations along a specified line, etc., there are some special cases where these hyperbolic motions have a very simple form:

- $T(z) = \bar{z}$ is a hyperbolic reflection in the Poincaré disk model.
- $T(z) = e^{i\theta}z$ is the rotation of angle θ about 0 in the Poincaré disk model (θ real).
- $T(z) = e^d z$ is a hyperbolic translation of distance $|d|$ along the positive imaginary axis in the Poincaré upper half-plane model (d real).

- $T(z) = z + b$ is a parallel displacement about the ideal point ∞ in the Poincaré upper half-plane model (b real).

We've also computed the Möbius transformations representing hyperbolic motions in a couple of more difficult cases:

- $T(z) = \frac{(e^d + 1)z + (e^d - 1)}{(e^d - 1)z + (e^d + 1)}$ is a hyperbolic translation of distance $|d|$ along the real axis in the Poincaré disk model (d real).
- $T(z) = \frac{(\cos \frac{\theta}{2})z + \sin \frac{\theta}{2}}{(-\sin \frac{\theta}{2})z + \cos \frac{\theta}{2}}$ is the hyperbolic rotation of angle θ about i in the Poincaré upper half-plane model.

Fortunately, we don't generally need to deal with more general Möbius transformations like these; by just using the three motions $z \mapsto e^{i\theta}z$, $z \mapsto e^d z$ and $z \mapsto z + b$ above, together with our isomorphism between the disk and upper half-plane models we were able to prove the following theorem:

Theorem. *Let \overrightarrow{AB} and \overrightarrow{CD} be Poincaré hyperbolic rays in the Poincaré disk model \mathbf{D} . Then there exists a Möbius transformation T such that $T(\mathbf{D}) = \mathbf{D}$ and $T(\overrightarrow{AB}) = \overrightarrow{CD}$.*

When we're proving something about hyperbolic geometry, this theorem allows us to assume that one of our P-rays is just the interval $[0, 1)$ of the positive real axis. For example, in proving the hyperbolic right triangle trig formulas in the Poincaré disk model, our first step was to apply a Möbius transformation to replace our triangle with one where $A = 0$ and C is positive real.

In the course of proving that SAS holds in the Poincaré disk model, we proved the following:

Theorem. *Two Poincaré triangle $\triangle ABC$ and $\triangle DEF$ in \mathbf{D} are congruent if and only if there there is a transformation T with $T(\mathbf{D}) = \mathbf{D}$ and $T(\triangle ABC) = \triangle DEF$, where T is either a Möbius transformation or the composition of a Möbius transformation with the hyperbolic reflection $z \mapsto \bar{z}$.*

The analogues of these theorems for \mathbf{H} are true as well; to prove them, we can just use the theorems for \mathbf{D} together with our standard isomorphism of \mathbf{H} with \mathbf{D} .

Arc lengths of paths and areas of regions in the hyperbolic plane can be computed using calculus. The hyperbolic length of an infinitesimal segment in one of the Poincaré models differs from the Euclidean length by a factor that depends on the Euclidean distance from the real axis in the upper half-plane model, or the distance from the center of the disk in the disk model. The formula for the upper half-plane is:

$$ds = \frac{|dz|}{y} = \frac{\sqrt{dx^2 + dy^2}}{y},$$

where $z = x + iy$. This means more precisely that if $\alpha(t) = x(t) + iy(t)$, $t \in [t_0, t_1]$ is a path in the upper half-plane, then its hyperbolic length is

$$\int_{\alpha} ds = \int_{t_0}^{t_1} \frac{|\alpha'(t)|}{y(t)} dt = \int_{t_0}^{t_1} \frac{\sqrt{[x'(t)]^2 + [y'(t)]^2}}{y(t)} dt.$$

For example, if $\alpha(t) = 0 + ti$, $t \in [1, a]$ is a path along the imaginary axis (a P-line), then we get

$$\int_{\alpha} ds = \int_1^a \frac{dt}{t} = \log a = d(i, ai).$$

Indeed, this explains where the $1/y$ is coming from: it is the derivative of the logarithm in the hyperbolic distance formula.⁴ Since expanding lengths by a factor of $1/y$ expands areas by a factor of $1/y^2$, we find that the infinitesimal hyperbolic area element is

$$dA = \frac{dx dy}{y^2}.$$

In the Poincaré disk model, using the Möbius transformation $z = i \frac{w+1}{-w+1}$ from \mathbf{D} to \mathbf{H} we can compute that for w real,

$$\begin{aligned} ds &= \frac{|dz|}{y} = \frac{\left| d \left(i \frac{w+1}{-w+1} \right) \right|}{\operatorname{Im} \left(i \frac{w+1}{-w+1} \right)} \\ &= \frac{\left| d \left(\frac{w+1}{-w+1} \right) \right|}{\frac{w+1}{-w+1}} \\ &= \frac{(-w+1)d(w+1) - (w+1)d(-w+1)}{(-w+1)^2} \\ &= \frac{\frac{w+1}{-w+1}}{(-w+1)} \\ &= \frac{(-w+1)dw + (w+1)dw}{(-w+1)(w+1)} \\ &= \frac{2dw}{1-w^2} \end{aligned}$$

Thus, by the rotational symmetry about 0 in the disk model, we get that the arc length in the disk model is

$$ds = \frac{2|dw|}{1-|w|^2}$$

and the infinitesimal area element is

$$dA = \frac{4d\tilde{x} d\tilde{y}}{(1-|w|^2)^2}$$

where $w = \tilde{x} + i\tilde{y}$.

⁴Here's a rather difficult exercise: prove that for $z_1, z_2 \in \mathbf{H}$, the path in \mathbf{H} from z_1 to z_2 of shortest hyperbolic arc length is the P-line segment between them. For the answer, take Math 401!