MATH 366: Midterm 2

Due Wednesday, March 31, 2010

Make sure to include your name and the honor code pledge. This exam is intended to be 3 hours long, but you may spend up to 4 hours on it. If you spend more than 3 hours, mark on your exam how far you are after 3 hours. The exam is closed-book and closed-notes, but you may use the list of axioms provided below:

(I-1) For every two distinct points P and Q there is a unique line ℓ passing through P and Q.

(I-2) For every line ℓ there exist two distinct points P and Q that lie on ℓ.

(I-3) There exist three distinct points that are not collinear.

(B-1) If $A * B * C$ then A, B, and C are three distinct collinear points and $C * B * A$.

(B-2) Given any two distinct points B and D, there exist points A, C, and E, lying on the line $\overleftrightarrow{BD}$, such that $A * B * D$, $B * C * D$, and $B * D * E$.

(B-3) If A, B, and C are three distinct collinear points, then one and only one of them is between the other two.

(B-4) For every line ℓ and points A, B, and C not lying on ℓ:

- (a) If A and B are on the same side of ℓ and if B and C are on the same side of ℓ, then A and C are on the same side of ℓ.
- (b) If A and B are on opposite sides of ℓ and if B and C are on opposite sides of ℓ, then A and C are on the same side of ℓ.

(C-1) If A and B are distinct points and if A' is any point, then for each ray r emanating from A', there is a unique point B' on r such that $AB \cong A'B'$.

(C-2) Congruence of segments is an equivalence relation.

(C-3) If $A * B * C$, $A' * B' * C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.

(C-4) Given any angle $\angle BAC$ and given any ray $\overrightarrow{A'B'}$, there is a unique ray $\overrightarrow{A'C'}$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\angle BAC \cong \angle B'A'C'$.

(C-5) Congruence of angles is an equivalence relation.

(C-6) [SAS] Suppose A, B, C are distinct, non-collinear points and A', B', C' are distinct, non-collinear points. Then if $AB \cong A'B'$, $AC \cong A'C'$, and $\angle BAC \cong \angle B'A'C'$, then $BC \cong B'C'$, $\angle ABC \cong \angle A'B'C'$, and $\angle ACB \cong \angle A'C'B'$.

1. In a Hilbert plane, suppose we are given points $A * B * C$ on a line ℓ and another point D not on ℓ so that $DC \perp \ell$. Prove that $AD > BD > CD$.

2. (a) Let A, B, C be distinct non-collinear points in a Hilbert plane. Suppose that the perpendicular bisectors of the segments AB and BC meet in a point P. Prove that the perpendicular bisector of AC also passes through P.

   (b) Let A, B, C be distinct non-collinear points in a Hilbert plane satisfying the Euclidean parallel axiom. Prove that there is a unique circle passing through A, B, and C.

   (c) Is it true in a hyperbolic plane that for every non-collinear points A, B, C there is a circle passing through all three? Explain.

3. Given four distinct complex numbers (we allow one to be ∞), $z_1, z_2, z_3, z_4 \in \hat{\mathbf{C}}$, their *cross-ratio* is the complex number

$$(z_1, z_2; z_3, z_4) := \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)},$$

   where we follow the usual convention based on the limit laws for "plugging in ∞" to the fraction, e.g.

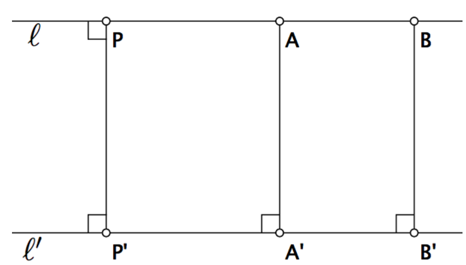
$$(z_1, \infty; z_3, z_4) = \frac{-(z_3 - z_4)}{(z_4 - z_1)}.$$

   (a) Let $z_1, z_2, z_3, z_4 \in \hat{\mathbf{C}}$ be distinct, and let T be any Möbius transformation. Prove that T preserves the cross-ratio, i.e. that $(z_1, z_2; z_3, z_4) = (T(z_1), T(z_2); T(z_3), T(z_4))$.

   (b) Prove that if $z_1, z_2, z_3, z \in \hat{\mathbf{C}}$ are distinct, then z is another point on the line or circle passing through z_1, z_2, and z_3 if and only if the cross-ratio $(z_1, z_2; z_3, z)$ is a real number.

4. Let PP' be a segment in a Hilbert plane $\mathcal{H}$, and let ℓ and ℓ' be the lines perpendicular to PP' at P and P' respectively. Let A and B be points of ℓ so that $P * A * B$, and drop perpendiculars AA' and BB' to ℓ' from A and B, respectively. Prove that

$$AA' < BB' \iff \mathcal{H} \text{ is semi-hyperbolic},$$
$$AA' \cong BB' \iff \mathcal{H} \text{ is semi-Euclidean, and}$$
$$AA' > BB' \iff \mathcal{H} \text{ is semi-elliptic.}$$



5. (a) State Archimedes' axiom.

   (b) State Aristotle's angle unboundedness axiom.

   (c) Prove that in a Hilbert plane satisfying Archimedes' axiom, given a ray $\overrightarrow{AB}$, a point P not on $\overleftrightarrow{AB}$, and an acute angle $\angle X$, there exists a point C on $\overrightarrow{AB}$ so that $\angle ACP < \angle X$.