MATH 366: Assignment 13

Due Friday, April 23, 2010

Hyperbolic geometry

1. Do exercises P19, P20 from chapter 7 in the textbook.

Complex numbers

2. Do exercise 71 from chapter 9 in the textbook. (You don't need to provide a general proof that your answers are right, but do include examples of fixed cycles (e.g. for rotations $z \mapsto e^{i\theta} z$ about 0 in the Poincaré disk model, the fixed cycles are the circles about 0; for the parallel displacements $z \mapsto z + b$ about ∞ in the Poincaré upper half-plane model, the fixed cycles are [insert what they are here], etc.).)

Hyperbolic trigonometry

3. Do problems 4, 13 from chapter 10 in the textbook.

Spherical Trigonometry

4. Consider the sphere

$$S_R^2 := \{(x, y, z) \in \mathbf{R}^3 : \quad x^2 + y^2 + z^2 = R^2\}$$

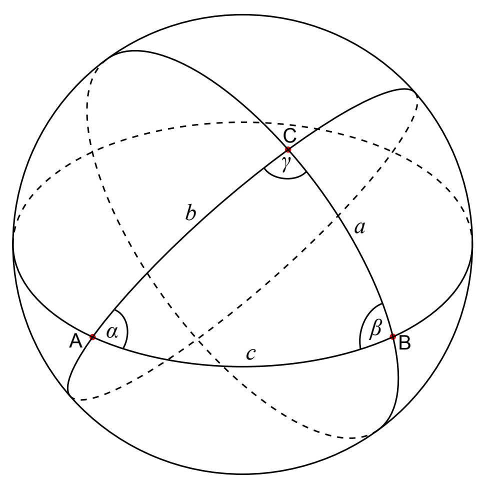
of radius R centered at the origin 0 in $\mathbf{R}^3$. We consider a spherical triangle ABC with vertices $A, B, C \in S_R^2$, angle measures α, β, γ (in radians), and side lengths a, b, c (see figure).

(a) Show that $B \cdot C = R^2 \cos \frac{a}{R}$, $A \cdot C = R^2 \cos \frac{b}{R}$, and $A \cdot B = R^2 \cos \frac{c}{R}$.

(b) Show that $\|A \times B\| = R^2 \sin \frac{c}{R}$ and that

$$\frac{A \times B}{\|A \times B\|} = \frac{A \times B}{R^2 \sin \frac{c}{R}}$$

is a unit normal vector at the origin to the plane spanned by 0, A, and B.



(c) The measure of the spherical angle α is, by definition, the measure of the dihedral angle between the plane spanned by 0, A, B and that spanned by 0, A, C. This is in turn equal to the angle between the normal vectors to these planes. Show then that:

$$(A \times B) \cdot (A \times C) = R^4 \sin \frac{b}{R} \sin \frac{c}{R} \cos \alpha.$$

(d) Prove the spherical law of cosines:

$$\cos\tfrac{a}{R} = \cos\tfrac{b}{R}\cos\tfrac{c}{R} + \sin\tfrac{b}{R}\sin\tfrac{c}{R}\cos\alpha,$$
$$\cos\tfrac{b}{R} = \cos\tfrac{a}{R}\cos\tfrac{c}{R} + \sin\tfrac{a}{R}\sin\tfrac{c}{R}\cos\beta,$$
$$\cos\tfrac{c}{R} = \cos\tfrac{a}{R}\cos\tfrac{b}{R} + \sin\tfrac{a}{R}\sin\tfrac{b}{R}\cos\gamma.$$

[Hint: Use parts (a) and (c) and the cross-product/dot-product identity $(W \times X)\cdot(Y \times Z) = (W \cdot Y)(X \cdot Z) - (W \cdot Z)(X \cdot Y)$.]

(e) Show that a spherical circle of spherical radius $r \le \frac{\pi}{2}R$ has circumference $2\pi R \sin\frac{r}{R}$ and area $2\pi R^2(1 - \cos\frac{r}{R})$. What happens when $\frac{\pi}{2}R < r \le \pi R$?

[Hint: feel free to use calculus to find the area.]

(f) (Extra Credit) Prove the spherical law of sines:

$$\frac{\sin\alpha}{\sin\frac{a}{R}} = \frac{\sin\beta}{\sin\frac{b}{R}} = \frac{\sin\gamma}{\sin\frac{c}{R}} = \frac{A \cdot (B \times C)}{R^3 \sin\frac{a}{R}\sin\frac{b}{R}\sin\frac{c}{R}}$$

[Hint: apply the cross product identity $X \times (Y \times Z) = (X \cdot Z)Y - (X \cdot Y)Z$ to expand $(A \times B) \times (A \times C)$ and then use some properties of the scalar triple product $X \cdot (Y \times Z) = \det(X, Y, Z)$.]

(g) (Extra Credit) Given a great circle ℓ on S^2_R, we say that the *poles* of ℓ are the two points where the line through 0 perpendicular to the plane spanned by ℓ meets S^2_R (e.g. the poles of the equator are the north and south poles).[1] Let A' be the pole of BC closest to A, B' be the pole of AC closest to B, and C' be the pole of AB closest to C. Let $\alpha', \beta', \gamma', a', b', c'$ be the corresponding angles and side lengths in the spherical triangle $A'B'C'$, which we call the *polar triangle* of ABC.

i. Show that ABC is the polar triangle of $A'B'C'$.

ii. Show that we have

$$\alpha' = \pi - \tfrac{a}{R}, \qquad \beta' = \pi - \tfrac{b}{R}, \qquad \gamma' = \pi - \tfrac{c}{R},$$
$$a' = R(\pi - \alpha), \qquad b' = R(\pi - \beta), \qquad c' = R(\pi - \gamma).$$

iii. Prove the "dual spherical law of cosines" for the original spherical triangle ABC:

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos\tfrac{a}{R},$$
$$\cos\beta = -\cos\alpha\cos\gamma + \sin\alpha\sin\gamma\cos\tfrac{b}{R},$$
$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\tfrac{c}{R}.$$

iv. Is there a "dual spherical law of sines"? Explain.

[1] Note that this terminology agrees with the use of the word "pole" in hyperbolic geometry, where the pole $P(\ell)$ is the ultra-ideal point with the property that all lines perpendicular to ℓ pass through $P(\ell)$. In elliptic geometry (where antipodal points of the sphere are identified), the pole of a "line" is just a single ordinary point.