MATH 366: Proofs of Propositions 2.1 and 2.2

Wednesday, January 20, 2010

Here are some sample proofs of the first two propositions from the textbook. When you write up proofs like this yourself, you need not write down the reasons for all the steps (as I did here in the footnotes), as writing down "by the definition of ..." every time you translate a definition into its meaning will make your proofs difficult to read. You should, however, be able to supply a justification for every single step, even if you don't write it down.

Proposition 2.1. *If l and m are distinct lines that are not parallel, then l and m have a unique point in common.*

Proof. Suppose l and m are distinct lines that are not parallel. Since l and m are not parallel, they certainly have a point in common[1]; let P be such a point. Let Q also be a point that lies on both l and m; we must show that $Q = P$.

Suppose that $Q \neq P$[2]. Then l and m are both lines passing through the distinct points P and Q, so by axiom I-1, we have $l = m$. But l and m are distinct, a contradiction. Thus $Q = P$[3], as desired, and l and m have a unique point in common. □

Proposition 2.2. *There exist three distinct lines that are not concurrent.*

Proof. By axiom I-3, there exist three distinct points that are not collinear[4]; call them P, Q, and R.

We claim first that the lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{QR}$, and $\overleftrightarrow{RP}$ are distinct. For suppose that $\overleftrightarrow{PQ} = \overleftrightarrow{QR}$[5]. Then P, Q, and R all lie on the line $\overleftrightarrow{PQ}$[6], but this is a contradiction since P, Q, and R are not collinear[7]. Thus $\overleftrightarrow{PQ} \neq \overleftrightarrow{QR}$[8] and similarly $\overleftrightarrow{QR} \neq \overleftrightarrow{RP}$[9] and $\overleftrightarrow{RP} \neq \overleftrightarrow{PQ}$, so we see that $\overleftrightarrow{PQ}$, $\overleftrightarrow{QR}$, and $\overleftrightarrow{RP}$ are distinct.

We show now that the lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{QR}$, and $\overleftrightarrow{RP}$ are not concurrent. For suppose that they are concurrent[10]; then there exists a point X which lies on all three[11]. The lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RP}$ are not parallel since P each lies on both[12]; however, the point X also lies on both $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RP}$, so by Proposition 2.1, $X = P$[13]. Likewise, the lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{QR}$ both pass through both Q and X, so that $X = Q$ by Proposition 2.1. But then $P = X = Q$ so that P and Q are not distinct, a contradiction. Thus $\overleftrightarrow{PQ}$, $\overleftrightarrow{QR}$, and $\overleftrightarrow{RP}$ are not concurrent[14]. □

---

[1] definition of parallel

[2] RAA hypothesis

[3] RAA conclusion

[4] definition of collinear

[5] RAA hypothesis

[6] as $\overleftrightarrow{PQ}$ is by definition the unique line passing through P and Q and $\overleftrightarrow{QR}$ is by definition the unique line passing through Q and R so if these lines are equal then $\overleftrightarrow{PQ}$ passes through P, Q, and R

[7] definition of collinear

[8] RAA conclusion

[9] By "similarly" we mean that we could go through the exact same argument as above, replacing every occurrence of $\overleftrightarrow{PQ}$ with $\overleftrightarrow{QR}$ and every occurrence of $\overleftrightarrow{QR}$ with $\overleftrightarrow{R}P$.

[10] RAA hypothesis

[11] definition of concurrent

[12] definition of parallel

[13] In order to apply Proposition 2.1, we need to know that the two lines are distinct, but we checked this earlier in the proof.

[14] RAA conclusion