MATH 366: Assignment 9

Due Friday, March 19, 2010

Neutral Geometry

1. Do exercise 15 and major exercises 5, 6, 7, 8 from chapter 4 in the textbook.

Complex Numbers

2. (a) Let $z_1, z_2, z_3 \in \mathbf{C}$ be distinct complex numbers. Show that

$$T(z) = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

defines a Möbius transformation for which $T(z_1) = 0$, $T(z_2) = 1$, and $T(z_3) = \infty$. What if one of the z_k is ∞?

(b) Show that the property you proved in (a) uniquely defines the Möbius transformation T, i.e. show that if S is a Möbius transformation with $S(z_1) = 0$, $S(z_2) = 1$, and $S(z_3) = \infty$, then $S = T$. [Hint: consider the fixed points of $S^{-1} \circ T$.]

(c) Show that for any distinct $z_1, z_2, z_3 \in \hat{\mathbf{C}}$ and any distinct $w_1, w_2, w_3 \in \hat{\mathbf{C}}$, there is a unique Möbius transformation U such that $U(z_1) = w_1$, $U(z_2) = w_2$, and $U(z_3) = w_3$.

(d) Show that for any three distinct points $z_1, z_2, z_3 \in \hat{\mathbf{C}}$ there is a unique line or circle passing through z_1, z_2, and z_3.

Extra Credit

3. In this problem, we compute the areas of spherical triangles. Let $S = \{(x, y, z) \in \mathbf{R}^3 : \ x^2 + y^2 + z^2 = 1\}$ be the unit sphere, and H_1, H_2, and H_3 be hemispheres of S whose bounding great circles are not concurrent.[1]. Then $H_1 \cap H_2 \cap H_3$ is the interior of a spherical triangle T. The intersections $H_1 \cap H_2$, $H_2 \cap H_3$, and $H_1 \cap H_3$ are lunes; let θ_{12}, θ_{23}, and θ_{13} be the radian measures of the angles at the vertices of these lunes (i.e. the angles of T).

(a) Show that the area of the lune $H_1 \cap H_2$ is $2\theta_{12}$.

(b) Show that $S - (H_1 \cup H_2 \cup H_3)$ is the spherical triangle antipodal to T (along with its interior), and hence has the same area as T.

(c) Show that the area of the triangle T is $\theta_{12} + \theta_{23} + \theta_{13} - \pi$. [Hint: $a(H_1 \cup H_2 \cup H_3) = a(H_1) + a(H_2) + a(H_3) - a(H_1 \cap H_2) - a(H_2 \cap H_3) - a(H_1 \cap H_3) + a(H_1 \cap H_2 \cap H_3)$,[2] where $a(X)$ denotes the area of $X \subset S$.]

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[1] See the extra credit problem from Assignment 6 for more details on how to describe hemispheres and great circles in coordinates using dot products.

[2] This formula is a special case of the *inclusion-exclusion principle*. You need not write a proof of it in your solution to this problem, but you should think about it if you've never seen it before; it may be helpful to consider a simpler version first:

$$a(X \cup Y) = a(X) + a(Y) - a(X \cap Y).$$

This same principle is frequently used in combinatorics with area replaced by the number of elements in a finite set.