1. Suppose that \( f, g \in k[x, y] \) where \( f \) is irreducible, \( f \nmid g \), and \( f \not\in k[x] \). Show that if we regard \( f \) and \( g \) as elements of \( k(x)[y] \), it is still the case that \( f \) is irreducible and \( f \nmid g \). [Hint: use Gauss’s Lemma.]

2. Suppose that \( C = V(f) \) is an irreducible affine plane curve of degree \( n \geq 2 \) with infinitely many points, where \( f = f_{n-1} + f_n \), with \( f_j \) homogeneous of degree \( j \). Show that \( C \) is rational. [Hint: show that if your parameterization were constant, then \( f \) would be reducible.]

3. Show that if \( k \) is an algebraically closed field and \( f \in k[x, y] \) is a homogeneous polynomial of degree \( n \), then \( f \) factors into a product of \( n \) homogeneous linear polynomials.

4. Suppose \( \text{char } k \neq 2, 3 \). Show that the affine plane curve \( y^2 = x^3 + px + q \) over \( k \) is singular if and only if the polynomial \( x^3 + px + q \) has a multiple root in \( k \).

5. Suppose that \( C \) is an irreducible affine plane curve of degree 3 that has infinitely many points. Show that \( C \) can have at most one singular point and that a singular point of \( C \) must have multiplicity two. What if \( C \) is reducible?

6. Show that if \( P_1 \) and \( P_2 \) are distinct points of an affine plane curve \( C \), then there exists a rational function \( f \in k(C) \) so that \( f(P_1) = 0 \) and \( f(P_2) = 1 \).