MATH 465/565: Homework 6

Due Friday, March 11, 2011

You should try to do as many of these problems as possible, but write up and turn in solutions to exactly six of them. We work over an algebraically closed field \( k \).

1. Prove that every rational map from \( \mathbb{P}^1 \) to \( \mathbb{P}^n \) is regular. [You should show this directly in this case, without using Theorem 1.1 as in the proof of Theorem 1.2.]

2. Let \( f : \mathbb{P}^2 \to \mathbb{P}^2 \) be the rational map defined by

\[
f([X_0, X_1, X_2]) = [X_1X_2, X_0X_2, X_0X_1].
\]

(a) Show that \( f \) defines a birational map from \( \mathbb{P}^2 \) to itself.

(b) At which points is \( f \) not regular? At which points is \( f^{-1} \) not regular?

(c) Find non-empty open sets \( U, V \subseteq \mathbb{P}^2 \) so that \( f \) restricts to an isomorphism \( U \to V \).

3. Prove that the Segre variety \( \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^N \), where \( N = (n+1)(m+1) - 1 \) is not contained in any hyperplane in \( \mathbb{P}^N \).

4. Let \( \Sigma_{n,m} = \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \) be the Segre image and let \( p_1 : \Sigma_{n,m} \to \mathbb{P}^n \) and \( p_2 : \Sigma_{n,m} \to \mathbb{P}^m \) be the functions obtained by composing the inverse function \( \sigma^{-1}_{n,m} : \Sigma_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m \) with the projections \( \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n \) and \( \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m \), respectively.

(a) Show that \( p_1 \) and \( p_2 \) are regular maps.

(b) Show that if \( W \) is any quasiprojective variety and \( f_1 : W \to \mathbb{P}^n \) and \( f_2 : W \to \mathbb{P}^m \) are regular maps, then there is a unique regular map \( f : W \to \Sigma_{n,m} \) such that \( f_1 = p_1 \circ f \) and \( f_2 = p_2 \circ f \).

5. Let \( X \subseteq \mathbb{P}^n \) and \( Y \subseteq \mathbb{P}^m \) be locally closed subsets (i.e. quasiprojective varieties).

(a) Show that the image under the Segre map \( \sigma_{n,m}(X \times Y) \subseteq \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subseteq \mathbb{P}^N \) is locally closed (i.e. \( \sigma_{n,m}(X \times Y) \) is a quasiprojective variety).

(b) Show that the functions \( p_1 : \sigma_{n,m}(X \times Y) \to X \) and \( p_2 : \sigma_{n,m}(X \times Y) \to Y \), defined as in the previous problem, are regular maps, and that if \( W \) is any quasiprojective variety and \( f_1 : W \to X \) and \( f_2 : W \to Y \) are regular maps, then there is a unique regular map \( f : W \to \sigma_{n,m}(X \times Y) \) such that \( f_1 = p_1 \circ f \) and \( f_2 = p_2 \circ f \).

(c) Suppose that \( X' \subseteq \mathbb{P}^{n'} \) and \( Y' \subseteq \mathbb{P}^{m'} \) are locally closed subsets so that \( X \cong X' \) and \( Y \cong Y' \). Show that the quasiprojective varieties \( \sigma_{n,m}(X \times Y) \subset \mathbb{P}^N \) and \( \sigma_{n',m'}(X' \times Y') \subset \mathbb{P}^{N'} \) are isomorphic.

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1 The is the universal property of the product; we say that \( \sigma_{n,m}(X \times Y) \) is the product of \( X \) and \( Y \) in the category of quasiprojective varieties.
6. Let $L$, $M$, and $N \subseteq \mathbb{P}^3$ be any pairwise skew (i.e. disjoint) lines. Show that, after a linear change of variables on $\mathbb{P}^3$, the union of the lines that meet $L$, $M$, and $N$ is the Segre surface $\Sigma_{1,1} \subseteq \mathbb{P}^3$.

7. Let $X$ be a quasi-projective variety and $U, V \subseteq X$ be affine open subsets. Show that $U \cap V$ is affine. [Hint: Consider $U \times V \subseteq X \times X$.]

8. Show that the twisted cubic curve $\nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$ can be realized as the intersection of $\Sigma_{1,2} \subseteq \mathbb{P}^5$ with a three-plane $\mathbb{P}^3 \subseteq \mathbb{P}^5$.

9. For this problem, let $k = \mathbb{C}$, the field of complex numbers. In this case, $\mathbb{A}^n$ has another standard topology, generated by open balls, which we will call the \textit{analytic topology} or the \textit{classical topology}.

(a) Show that the classical topology on $\mathbb{A}^n$ is finer than the Zariski topology (i.e. show that every Zariski closed subset is also closed in the classical topology).

(b) Let $X \subseteq \mathbb{A}^n$ be closed in the Zariski topology. Prove that $\mathbb{A}^n \setminus X$ is dense in the classical topology.

(c) Describe what the classical topology should be on $\mathbb{P}^n$. Show that $\mathbb{P}^n$ is compact in the classical topology.

(d) Show that every regular function on an irreducible projective variety (over $k = \mathbb{C}$) is constant.

10. (Extra Credit) Let $\ell \subseteq \Sigma_{n,m} \subset \mathbb{P}^N$ be a line contained in the Segre image. Show that either $\ell$ is the image under $\sigma_{n,m}$ of $\ell_1 \times \{y_2\}$ for some line $\ell_1 \subseteq \mathbb{P}^n$ and point $y_2 \in \mathbb{P}^m$, or $\ell$ is the image of $\{x_1\} \times \ell_2$ for some point $x_1 \in \mathbb{P}^n$ and line $\ell_2 \subseteq \mathbb{P}^m$. 

2