MATH 465/565: Homework 9

Due Friday, April 8, 2011

We work over an algebraically closed field $k$.

1. Let $H \subseteq \mathbb{P}^n$ be a hyperplane, $X \subseteq H$ an irreducible closed subset, and $y$ be any point in $\mathbb{P}^n \setminus H$. Let $Y \subseteq \mathbb{P}^n$ be the union of all the lines connecting $y$ with some point $x \in X$. (We call $Y$ the cone over $X$ with vertex $y$.) Show that $Y$ is an irreducible projective variety and $\dim Y = \dim X + 1$. [Hint: pick a convenient coordinate system.]

2. (a) Prove that any finite set $S \subseteq \mathbb{A}^2$ is the zero locus of two polynomials. [Hint: We can take the first polynomial to be a polynomial in $x$ alone.]

(b) Prove that any finite set $S \subseteq \mathbb{P}^2$ is the zero locus of two homogeneous polynomials.

(c) Find an example of a finite set $S \subseteq \mathbb{P}^2$ so that the homogeneous ideal $I(S)$ of $S$ cannot be generated by two homogeneous polynomials.\[1\]

3. Let $X \subseteq \mathbb{P}^n$ be closed, $I = I(X)$ its homogeneous ideal. Let $R_d = k[X_0, \ldots, X_n]$ be the vector space of degree $d$ homogeneous polynomials in $n + 1$ variables, and let $I_d = I \cap R_d$. The Hilbert function of $X$ in $\mathbb{P}^n$ is

$$h_X(d) = \dim(R_d/I_d) = \dim R_d - \dim I_d.$$ 

(a) Determine $h_X$ when $X \subseteq \mathbb{P}^2$ consists of three points.\[2\]

(b) Determine $h_X$ when $X \subseteq \mathbb{P}^2$ consists of four points.\[2\]

4. Given a ring $R$, an ascending chain of prime ideals of length $n$ consists of $n + 1$ prime ideals

$$p_0 \not\subseteq p_1 \not\subseteq \cdots \not\subseteq p_n$$

of $R$, totally ordered by inclusion. The Krull dimension of $R$ is the supremum of the lengths of all the ascending chains of prime ideals in $R$.

(a) If $X$ is an affine variety, show that the Krull dimension of its coordinate ring $k[X]$ is equal to the dimension of $X$.

(b) Compute the Krull dimension of the ring $\mathbb{Z}[x]$.

5. Show that the linear groups $GL_n(k)$, $SL_n(k)$, and $PGL_n(k)$ have natural structures as quasiprojective varieties and compute their dimensions.

\[1\] A projective variety $X \subseteq \mathbb{P}^n$ of dimension $m$ is a complete intersection if its homogenous ideal $I(X)$ is generated by exactly $n - m$ elements; $X$ is a set-theoretic complete intersection if $X$ is the zero locus of $n - m$ homogeneous polynomials. This $S$ provides an example of a set-theoretic complete intersection which is not a complete intersection.

\[2\] There may be several different cases with different answers depending on how the points are arranged. You should find $h_X$ for each case.
6. Given $n + 1$ homogeneous polynomials $F_0, \ldots, F_n \in k[X_0, \ldots, X_n]$ of degrees $d_0, \ldots, d_n$ in $n + 1$ variables, we may consider their common zero locus. Let $R_d = k[X_0, \ldots, X_n]_d$, the vector space of homogeneous polynomials in $n + 1$ variables, and set

$$
\Gamma = \{([F_0], \ldots, [F_n], x) \in \mathbb{P}(R_{d_0}) \times \cdots \times \mathbb{P}(R_{d_n}) \times \mathbb{P}^n : \ F_0(x) = F_1(x) = \cdots = F_n(x) = 0\}.
$$

(a) Show that $\Gamma$ is a closed subset of $\mathbb{P}(R_{d_0}) \times \cdots \times \mathbb{P}(R_{d_n}) \times \mathbb{P}^n$.

(b) By considering the projection map $\psi: \Gamma \to \mathbb{P}^n$, show that

$$
\dim \Gamma = n + \sum_i \left(\dim \mathbb{P}(R_{d_i}) - 1\right).
$$

(c) By considering the projection map $\varphi: \Gamma \to \prod_i \mathbb{P}(R_{d_i})$, show that

$$
\dim \varphi(\Gamma) = \dim \Gamma = \dim \prod_i \mathbb{P}(R_{d_i}) - 1.
$$

(d) Show that there exists a multihomogeneous polynomial $R$ in the coefficients of the $F_i$ such that $R = 0$ if and only if the system $F_0(x) = F_1(x) = \cdots = F_n(x) = 0$ has a non-zero solution.

(e) What is the polynomial $R$ in the case $d_0 = \cdots = d_n = 1$?