

MATH 499: Homework 10

November 5, 2009

1. Calculate the Tjurina numbers for the following singular curves (each has a single singularity at the origin). The first few are some of the curves from the beginning of the course.
 - (a) $f(x, y) = y^2 - x^3 - x^2$
 - (b) $f(x, y) = y^2 - x^3$
 - (c) $f(x, y) = x^4 + y^4 - x^2$
 - (d) $f(x, y) = x^2y + xy^2 - x^4 - y^4$
 - (e) $f(x, y) = x^3y - xy^2$
 - (f) $f(x, y) = y^3 - x^4$
 - (g) $f(x, y) = x^3 + y^3 - x^2y$
 - (h) $f(x, y) = y^3 - x^7 + x^5y$
2. Calculate the Tjurina number of $f(x, y) = x^p + y^q$ for positive integers p and q .
3. Is there any relation between the Tjurina number and the multiplicity? (e.g. is it possible for the Tjurina number to be large and the multiplicity to be small or vice versa?) Explain.
4. We mentioned an alternative definition for the resultant: given two monic polynomials $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ and $g(t) = t^m + b_{m-1}t^{m-1} + \cdots + b_0$, by the fundamental theorem of algebra, we can factor them over the complex numbers as

$$\begin{aligned}f(t) &= (t - \alpha_1)(t - \alpha_2)(t - \alpha_3) \cdots (t - \alpha_{n-1})(t - \alpha_n), \\g(t) &= (t - \beta_1)(t - \beta_2) \cdots (t - \beta_m).\end{aligned}$$

Then if we form the product

$$R(f, g, t) = \prod_{j=1}^n \prod_{k=1}^m (\alpha_j - \beta_k)$$

which is clearly a degree nm polynomial in the α_j, β_k . On the other hand, our original definition of the resultant $\text{Res}(f, g, t)$ is a polynomial in the coefficients a_j, b_k , but we can express the coefficients in terms of the roots by

$$\begin{aligned}a_{n-r} &= (-1)^r \sum_{i_1 < i_2 < \cdots < i_r} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_r} \\b_{m-s} &= (-1)^s \sum_{i_1 < i_2 < \cdots < i_s} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_s}\end{aligned}$$

which allows us to express the original resultant $\text{Res}(f, g, t)$ as a polynomial in the α_j, β_k . Show that this is also a polynomial of degree nm in the α_j, β_k .