MATH 499: Homework 4

September 17, 2009

1. In each part, determine whether the polynomial $f \in \mathbf{R}[x]$ is in the given ideal $I \subseteq R[x]$. Notice that determining if f lies in the ideal $\langle g \rangle$ is equivalent to determining if g divides f. How do we use the same idea in (c) and (d), where $I = \langle g_1, g_2 \rangle$?

   (a) $f(x) = x^2 - 3x + 2,$ $\quad I = \langle x - 2 \rangle$

   (b) $f(x) = x^5 - 4x + 1,$ $\quad I = \langle x^3 - x^2 + x \rangle$

   (c) $f(x) = x^2 - 4x + 4,$ $\quad I = \langle x^4 - 6x^2 + 12x - 8, 2x^3 - 10x^2 + 16x - 8 \rangle$

   (d) $f(x) = x^3 - 1,$ $\quad I = \langle x^9 - 1, x^5 + x^3 - x^2 - 1 \rangle$

2. Find an ideal $I \subset \mathbf{R}[x]$ in which every element $f \in I$ is divisible by x, but such that $x \notin I$.

3. (a) Show that $\langle x - y^2, xy, y^2 \rangle = \langle x, y^2 \rangle$.

   (b) Is $\langle x - y^2, xy \rangle = \langle x^2, xy \rangle$?

4. Rewrite each of the following polynomials, ordering the terms first with the lex order, then the graded lex order, and finally the graded reverse lex order, provided that $x > y > z$.

   (a) $f(x, y, z) = 2x + 3y + z + x^2 - z^2 + x^3$

   (b) $2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4$

   (c) $7x^2y^4z - 2xy^6 + x^2y^2$

5. Ideals make sense in the ring of integers $\mathbf{Z}$ just as they do in polynomial rings like $\mathbf{R}[x]$. For example, in $\mathbf{Z}$ the ideal $I = \langle a, b \rangle$ consists of all integers $xa + yb$ for $x, y \in \mathbf{Z}$.

   (a) Is 10 in the ideal $I = \langle 3 \rangle$?

   (b) Is 2 in the ideal $I = \langle 5, 8 \rangle$?

   (c) Is -6 in the ideal $I = \langle 12, 22 \rangle$?

   (d) Is 3 in the ideal $I = \langle 68, 173 \rangle$?