# MATH 499: Homework 7

October 15, 2009

1. Let $I \subseteq \mathbf{R}[x_1, \ldots, x_n]$ be an ideal.

   (a) The *lth elimination ideal* is defined to be the set
   $$I_l = I \cap \mathbf{R}[x_{l+1}, \ldots, x_n].$$
   Prove that $I_l$ is indeed an ideal of $\mathbf{R}[x_{l+1}, \ldots, x_n]$.[1]

   (b) Is $I_l$ an ideal of $\mathbf{R}[x_1, \ldots, x_n]$?

   (c) Prove that the ideal $I_{l+1} \subseteq \mathbf{R}[x_{l+2}, \ldots x_n]$ is the first elimination ideal of $I_l \subseteq \mathbf{R}[x_{l+1}, \ldots, x_n]$, i.e. that $(I_l)_1 = I_{l+1}$.

2. Consider the system of equations
   $$x^2 + 2y^2 = 3$$
   $$x^2 + xy + y^2 = 3.$$

   (a) Let $I = \langle x^2 + 2y^2 - 3, x^2 + xy + y^2 - 3 \rangle$. Find Gröbner bases for $I \cap \mathbf{Q}[x]$ and $I \cap \mathbf{Q}[y]$.

   (b) Find all solutions of the system over the complex numbers $\mathbf{C}$.

   (c) Which of the solutions are rational; that is, which solutions like in $\mathbf{Q}^2$?

3. Find all rational solutions $(x, y) \in \mathbf{Q}^2$ and all complex solutions $(x, y) \in \mathbf{C}^2$ of the system
   $$x^2 + 2y^2 = 2$$
   $$x^2 + xy + y^2 = 2.$$

4. Consider the system of equations
   $$t^2 + x^2 + y^2 + z^2 = 0$$
   $$t^2 + 2x^2 - xy - z^2 = 0$$
   $$t + y^3 - z^3 = 0.$$
   Suppose we want to eliminate $t$. Let
   $$I = \langle t^2 + x^2 + y^2 + z^2, t^2 + 2x^2 - xy - z^2, t + y^3 - z^3 \rangle \subset \mathbf{R}[t, x, y, z]$$
   be the corresponding ideal.

   (a) Use lex order with $t > x > y > z$ to compute a Gröbner basis for $I$, and then find a bass for $I \cap \mathbf{R}[x, y, z]$. You should get four generators, one of which has total degree 12.

   (b) Compute a reduced Gröbner basis for the ideal $I \cap \mathbf{R}[x, y, z]$ in grevlex order. This time, you should get a set of two generators.

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[1] An *ideal* $I$ of $R = \mathbf{R}[x_{l+1}, \ldots, x_n]$ is a subset of $R$ such that:

- $0 \in I$,
- if $f, g \in I$ then $f + g \in I$, and
- if $f \in I$ and $g \in R$, then $gf \in I$.