MATH 499: Homework 9

October 29, 2009

1. If $f, g \in \mathbf{C}[x]$ are non-constant polynomials over the complex numbers, show that f and g have a common root in $\mathbf{C}$ if and only if $\mathrm{Res}(f, g, x) = 0$.

   [Hint: Use the "fundamental theorem of algebra": the fact that any non-constant polynomial in $\mathbf{C}[x]$ factors completely into linear factors.]

2. If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in k[x]$, where $a_n \neq 0$ and $n > 0$, then the *discriminant* of f is defined to be

$$\mathrm{disc}(f) = \frac{(-1)^{n(n-1)/2}}{a_n}\mathrm{Res}(f, f', x)$$

   Prove that f has a multiple factor (that is, f is divisible by g^2 for some non-constant $g \in k[x]$) if and only if $\mathrm{disc}(f) = 0$.

3. (a) Compute the discriminant of the polynomial $x^2 + bx + c$ for $b, c \in k$

   (b) Compute the discriminant of the polynomial $x^3 + px + q$ for $p, q \in k$.

4. An alternative definition of the discriminant of a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbf{C}[x]$ of degree n is

$$D(f) = a_n^{2n-2} \prod_{i<j} (\alpha_i - \alpha_j)^2$$

   where $\alpha_1, \ldots, \alpha_n$ are the roots of f, counted with multiplicities, i.e. $f(x) = a_n \prod_i (x - \alpha_i)$.

   (a) Show that $D(f)$ doesn't depend on the choice of ordering on the roots α_i.

   (b) Show that $\prod_{i<j} (\alpha_i - \alpha_j)^2 = (-1)^{n(n-1)/2} \prod_{i \neq j} (\alpha_i - \alpha_j)$.

5. For $f, g \in \mathbf{Z}[x]$, show that $\mathrm{Res}(f, g, x) \in \mathbf{Z}$.

6. Let $f, g \in k[x]$ have degrees n and m. Show that if the $(m + n) \times (m + n)$ Sylvester matrix has rank $m + n - 1$, then f and g share a common linear factor, but not a common quadratic factor.