

# MATH 499: Notes

August 27, 2009

In class we saw that a point  $p$  is a *singular point* of a plane curve  $f(x, y) = 0$  if and only if  $f(p) = 0$  so that  $p$  is on the curve, and

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = 0.$$

In single variable calculus, in studying points where the first derivative of a function  $g(x)$  is zero, it is often helpful to study the higher derivatives. For example, if  $g'(a) = 0$  and  $g''(a) > 0$ , then  $g$  has a local minimum at  $a$ . More generally, if  $g$  is a sufficiently nice function<sup>1</sup>, then  $g$  is represented near  $a$  by its Taylor series centered at  $a$ ,

$$g(x) = g(a) + g'(a)(x - a) + \frac{g''(a)}{2!}(x - a)^2 + \frac{g^{(3)}(a)}{3!}(x - a)^3 + \dots$$

and the behavior of  $g$  very close to  $a$  is completely determined (in a sense that we will make more precise in the future) by the first non-constant term of the Taylor series.

In studying functions of several variables, when  $\nabla f|_p = 0$ , it also makes sense to look at higher derivatives of  $f$  at  $p$ . In fact, Taylor series work fine in several variables. The idea is the same as it is in the one variable case: we find a polynomial of degree  $n$  in several variables all of whose partial derivatives up to order  $n$  agree with those of the function  $f$ . Letting  $n \rightarrow \infty$ , we get an infinite power series representation in several variables for  $f(x, y)$  centered at  $p = (x_0, y_0)$  that looks like:

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<sup>1</sup>The technical term is “real analytic,” which just means represented by its Taylor series near each point. Essentially all infinitely differentiable functions one encounters in practice are real analytic. Probably the simplest example of a smooth function that isn’t real analytic is something like

$$g(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

You can check that this function is infinitely differentiable, but  $g(x) \rightarrow 0$  so fast as  $x \rightarrow 0$  that  $g^{(n)}(0) = 0$  for all  $n$ . Thus its Taylor series centered at  $x = 0$  is identically zero, and is not equal to the function except at zero.

Fortunately, we’ll be working with polynomials and power series directly, so we won’t need to worry at all about pathological functions like this. In fact, we won’t even really have to worry about convergence of the power series we deal with!

$$\begin{aligned}
f(x, y) &= f(p) + \frac{\partial f}{\partial x} \Big|_p (x - x_0) + \frac{\partial f}{\partial y} \Big|_p (y - y_0) \\
&\quad + \frac{1}{2!} \cdot \frac{\partial^2 f}{\partial x^2} \Big|_p (x - x_0)^2 + \frac{2}{2!} \cdot \frac{\partial^2 f}{\partial x \partial y} \Big|_p (x - x_0)(y - y_0) + \frac{1}{2!} \cdot \frac{\partial^2 f}{\partial y^2} \Big|_p (y - y_0)^2 \\
&\quad + \frac{1}{3!} \cdot \frac{\partial^3 f}{\partial x^3} \Big|_p (x - x_0)^3 + \frac{3}{3!} \cdot \frac{\partial^3 f}{\partial x^2 \partial y} \Big|_p (x - x_0)^2(y - y_0) + \frac{3}{3!} \cdot \frac{\partial^3 f}{\partial x \partial y^2} \Big|_p (x - x_0)(y - y_0)^2 \\
&\quad\quad\quad + \frac{1}{3!} \cdot \frac{\partial^3 f}{\partial y^3} \Big|_p (y - y_0)^3 + \dots \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \frac{\partial^n f}{\partial x^k \partial y^{n-k}} \Big|_p (x - x_0)^k (y - y_0)^{n-k}.
\end{aligned}$$

If  $f$  is a polynomial to start with, the resulting ‘‘Taylor series’’ will have only finitely many terms. For example, if we expand  $f(x, y) = y^2 - x^3 + 12x - 16$  around the singular point  $(2, 0)$ , we get

$$f(x, y) = y^2 - 6(x - 2)^2 - (x - 2)^3.$$

A computer algebra system can compute these Taylor series expansions for us. For example, the Mathematica command `Series[x^2 y + x y^2, {x, 3, 10}, {y, -1, 10}]` produces the output

$$\begin{aligned}
&(-6 + 3(y + 1) + 3(y + 1)^2 + O((y + 1)^{11})) + (-5 + 4(y + 1) + (y + 1)^2 + O((y + 1)^{11})) (x - 3) \\
&\quad + (-1 + (y + 1) + O((y + 1)^{11})) (x - 3)^2 + O((x - 3)^{11}),
\end{aligned}$$

where here we are asking Mathematica to write  $x^2y + xy^2$  as a series in  $x - 3$  whose coefficients are series in  $y + 1$ . The 10s in the command are telling Mathematica to only compute the terms of the series up to degree 10, and the  $O((y + 1)^{11})$  and  $O((x - 3)^{11})$  are reminding us that as far as Mathematica knows, there may be other terms in the Taylor series involving higher powers of  $y + 1$  or  $x - 3$ . On the other hand, we know that really we’re dealing with a polynomial of degree 3, so higher degree terms can’t be there. Grouping the terms from the Mathematica output, we get:

$$\begin{aligned}
x^2y + xy^2 &= -6 - 5(x - 3) + 3(y + 1) \\
&\quad - (x - 3)^2 + (x - 3)(y + 1) + 3(y + 1)^2 \\
&\quad + (x - 3)^2(y + 1) + (x - 3)(y + 1)^2.
\end{aligned}$$

David has pointed out that you can use the Mathematica command

$$\text{Normal}[\text{Series}[x^2 y + x y^2, \{x, 3, 10\}, \{y, -1, 10\}]]$$

to put it into a nicer form and remove the  $O((x - 3)^{11})$  terms.