We’ve now seen plane curves with various singularities and given a few types of them names, although we haven’t give any precise mathematical definitions of these yet.

One simple question we should expect to be able to answer using calculus is what are the tangent lines to the branch or branches of the curve at the singular point? (In the node example above, we can see two distinct tangent lines in the picture, whereas in the cusp and tacnode cases, there should only be one tangent line at the singular point.)

To simplify our computations, we may as well first make a change of coordinates so that the singular point is at the origin (we can certainly do this with an affine change of variables—see the homework). Consider then a plane curve $V(f)$, where $f$ is a polynomial with $f(0,0) = 0$.

Of course, if the origin were a smooth point, this would be easy: there is only one tangent line, defined by

$$\frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y = 0.$$ 

When the origin is singular, both first partial derivatives are zero and we need to look at higher order terms to find the tangent lines. In particular, let us write $f = f_m + f_{m+1} + \ldots + f_n$ where $f_k$ is the degree $k$ part of the polynomial $f$; here $n = \deg f$ and $f_m \neq 0$ is the non-zero piece of $f$ of smallest degree.

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1The notation $V(f)$ will be used to mean $\{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\}$. More generally, an algebraic variety over the real numbers $\mathbb{R}$ will be the locus

$$V(f_1, f_2, \ldots, f_k) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : f_1(x_1, \ldots, x_n) = f_2(x_1, \ldots, x_n) = \cdots = f_k(x_1, \ldots, x_n) = 0\}$$

of common zeros of finitely many polynomials in $n$ variables $f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n]$.  

Proposition. If \( L = V(ax + by) \) is a tangent line to \( V(f) \) at \( p = (0,0) \), then \( ax + by \) is a factor of \( f_m \).

Proof. Let \((x(t), y(t))\) for \( t \in [0,1)\) be a local parametrization of a branch of \( X = V(f) \) near the origin, with \( x(0) = y(0) = 0 \). We assume for convenience that the branch does not have a vertical tangent, i.e. that \( y(t)/x(t) \) is bounded as \( t \to 0 \). Then the slope of the tangent line to this branch of \( X \) at the origin should be \( \lim_{t \to 0^+} \frac{y(t)}{x(t)} \). We write

\[
\frac{f_m(x,y)}{x^m} = a_0 x^m + a_1 x^{m-1} y + \cdots + a_{m-1} x y^{m-1} + a_m y^m
\]

so that for \((x,y) = (x(t), y(t))\) with \( t \) very small,

\[
0 = f(x,y) = a_0 x^m + a_1 x^{m-1} y + \cdots + a_{m-1} x y^{m-1} + a_m y^m + f_{m+1}(x,y) + \cdots + f_n(x,y),
\]

and dividing through by \( x^m \), we get

\[
0 = \frac{f(x,y)}{x^m} = a_0 + a_1 \left( \frac{y(t)}{x(t)} \right) + \cdots + a_{m-1} \left( \frac{y(t)}{x(t)} \right)^{m-1} + a_m \left( \frac{y(t)}{x(t)} \right)^m + \frac{f_{m+1}(x,y)}{x^m} + \cdots + \frac{f_n(x,y)}{x^m}.
\]

Now, since \( x(t) \to 0 \) and \( y(t) \to 0 \) as \( t \to 0^+ \) and \( y(t)/x(t) \) stays bounded, \( \lim_{t \to 0^+} \frac{x(t)^{j+1}y(t)^{m+1}}{x(t)^m} = 0 \) for \( i > 0 \) and we find that

\[
\lim_{t \to 0^+} \frac{f_{m+1}(x,y)}{x^m} + \cdots + \frac{f_n(x,y)}{x^m} = 0.
\]

This implies that

\[
\lim_{t \to 0^+} a_0 + a_1 \left( \frac{y(t)}{x(t)} \right) + \cdots + a_{m-1} \left( \frac{y(t)}{x(t)} \right)^{m-1} + a_m \left( \frac{y(t)}{x(t)} \right)^m = 0,
\]

so that the slope \( c \) of the tangent line is a root of the polynomial \( a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + a_m z^m \).

Equivalently, \( z - c \) is a factor of \( a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + a_m z^m \) and \( y - cx \) is a factor of \( f_m = a_0 x^m + a_1 x^{m-1} y + \cdots + a_{m-1} x y^{m-1} + a_m y^m \) as desired (Why?).

If we had been dealing with a vertical tangent, then we could have divided through by \( y^m \) instead, and we would find that

\[
\lim_{t \to 0^+} a_0 \left( \frac{x(t)}{y(t)} \right)^m + a_1 \left( \frac{x(t)}{y(t)} \right)^{m-1} + \cdots + a_{m-1} \left( \frac{x(t)}{y(t)} \right) + a_m = 0,
\]

which implies that 0 must be a root of the polynomial \( a_0 w^m + a_1 w^{m-1} + \cdots + a_{m-1} w + a_m \) so that \( x \) is a factor of the polynomial \( f_m = a_0 x^m + a_1 x^{m-1} y + \cdots + a_{m-1} x y^{m-1} + a_m y^m \).

We would very much like the converse of this proposition to be true, i.e. that if \( ax + by \) is a factor of \( f_m \), then near the origin there is a branch of the curve \( V(f) \) having \( V(ax + by) \) as its tangent line at the origin. It has been suggested though that one of our homework examples shows that this is not the case, at least not over \( \mathbb{R} \).

We call that smallest degree \( m \) of a non-zero term of \( f \) the multiplicity of the curve \( X = V(f) \) at the origin. Since \( V(f_m) \) is often the union of the tangent lines to branches of the curve \( X \) at the origin (and at least always contains those lines) we’ll give it a name: we call \( V(f_m) \) the tangent cone of \( X \) at the origin and denote it \( TC_{(0,0)}(X) \).
We said that we could move a singular point of \( X = V(f) \) to the origin by a change of coordinates, but we could also have done everything with Taylor expansions centered at any point \( p = (a, b) \): we set \( f_k \) to be the degree \( k \) part of \( f \) regarded as a polynomial in \( x - a \) and \( y - b \), as can be computed by Taylor expansion

\[
f_k = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{\partial^n f}{\partial x^j \partial y^{k-j}} \bigg|_p (x - a)^j (y - b)^{k-j}.
\]

We can then define the multiplicity \( m \) as before as the the smallest number so that \( f_m \neq 0 \), and define the tangent cone to \( X \) at \( p \) to be \( TC_p(X) = V(f_m) \). One can check that this agrees with what we’d get by instead translating the curve \( X \) so that \( p \) goes to the origin, computing the tangent cone at the origin as above and then translating back.

One might wonder where the term “cone” is coming from here given that there is no cone in the traditional sense in sight. The answer is that the term “cone” is often used more generally to refer to any locus traced out by lines through some fixed point (the vertex of the “cone”). Here, the “base” of the tangent cone of our plane curve might be regarded as finitely many points, one for each tangent direction. While this can never really look much like a more familiar cone in the case of plane curves, it is possible for a “tangent cone” to a singular point on a surface to actually be a “cone” in the original sense (see Figure 2).

Figure 2: Tangent cone at a surface singularity

Exercise. Show that if \( f(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n] \) is a homogeneous polynomial (all of its
terms have the same degree), then its zero locus $V(f)$ is a cone with vertex $(0, \ldots, 0)$, i.e. if $(c_1, \ldots, c_n) \in V(f)$ then $(\lambda c_1, \ldots, \lambda c_n) \in V(f)$ for all $\lambda \in \mathbb{R}$.

If we were working over the complex numbers, the converse would be true as well: if $V(f)$ is a cone with vertex at the origin, then the polynomial $f$ must be homogeneous. Why doesn’t this work over the reals? Can you find a counterexample?

There’s another (closely related) notion of “multiplicity,” namely the multiplicity of intersection of a curve and a line. Given a plane curve $X = V(f)$ and a line $L$ through a point $p = (a, b) \in X$, we can define the multiplicity of their intersection as follows: we choose a linear parametrization of $L$ as $\alpha(t) = (a, b) + t(c, d)$ so that $\alpha(0) = p$. The composition $f(\alpha(t))$ is a polynomial in $t$ with 0 as a root. The intersection multiplicity of $X$ and $L$ at $p$ is defined to be the multiplicity of 0 as a root of $f(\alpha(t))$, i.e. the power of $t$ in the factorization of $f(\alpha(t))$.

Suppose that $p$ is a smooth point of $X$. Then we have

$$f(\alpha(t)) = f(a + tc, b + td) = \left. \frac{\partial f}{\partial x} \right|_p (a + tc - a) + \left. \frac{\partial f}{\partial y} \right|_p (b + td - b) + f_2(a + tc, b + td) + \cdots + f_n(a + tc, b + td)$$

$$= t \left( c \left. \frac{\partial f}{\partial x} \right|_p + d \left. \frac{\partial f}{\partial y} \right|_p \right) + t^2 F_2(c, d) + t^3 F_3(c, d) + \cdots + t^n F_n(c, d),$$

where $F_k(c, d) = f_k(a + c, b + d)$.\(^1\) We see then that the intersection multiplicity of $X$ with $L$ at the smooth point $p$ is 1 except when $c \left. \frac{\partial f}{\partial x} \right|_p + d \left. \frac{\partial f}{\partial y} \right|_p = 0$, i.e. when $L$ is the tangent line to $X$ at $p$, in which case the intersection multiplicity is at least 2, and we would need to look at the higher order terms to compute it exactly.

The same computation in the case where $p$ is singular shows that the intersection multiplicity of $X$ with $L$ at $p$ is at least 2 for every line $L$ through $p$.

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\(^1\) The fact that the $F_k(c, d)$ in our expression above are homogeneous polynomials in $c$ and $d$ also helps show that the intersection multiplicity does not depend on our choice of linear parametrization $\alpha$ for $L$. 