MATH 499: Notes

October 29, 2009

Last time we claimed (without proof) that the resultant of two monic polynomials could be defined in terms of the roots. Of course, we can do the same with polynomials which aren't monic: if $f(t) = a_n t^n + \cdots + a_0$ and $g(t) = b_m t^m + \cdots + b_0$ are polynomials with complex coefficients, then we may write

$$\begin{aligned} f(t) &= a_n(t-\alpha_1)(t-\alpha_2)(t-\alpha_3)\cdots(t-\alpha_{n-1})(t-\alpha_n) \\ g(t) &= b_m(t-\beta_1)(t-\beta_2)\cdots(t-\beta_m) \end{aligned}$$

where the α_i are the roots of f and the β_j are the roots of g, counted with multiplicities. In the Sylvester matrix, replacing f and g by the monic polynomials with the same roots corresponds to dividing the first m columns by a_n and dividing the first n columns by b_m. Thus the general formula for the resultant in terms of the roots and a_n and b_m should be

$$R(f,g,t) = a_n^m b_m^n \prod_{i=1}^{n}\prod_{j=1}^{m}(\alpha_i - \beta_j) = a_n^m \prod_{i=1}^{n} g(\alpha_i)$$

Now, consider the case of the discriminant, i.e. take $g = f'$. We have

$$f'(t) = a_n \sum_k \prod_{j\neq k}(t-\alpha_j)$$

by the product rule. Thus we have

$$\begin{aligned} R(f,f',t) &= a_n^{n-1}\prod_i f'(\alpha_i) \\ &= a_n^{n-1}\prod_i\left(a_n\sum_k\prod_{j\neq k}(\alpha_i-\alpha_j)\right) \\ &= a_n^{2n-1}\prod_i\left(\sum_k\prod_{j\neq k}(\alpha_i-\alpha_j)\right) \\ &= a_n^{2n-1}\prod_i\left(\prod_{j\neq i}(\alpha_i-\alpha_j)\right) && \text{since if } k\neq i \text{ then } \prod_{j\neq k}(\alpha_i-\alpha_j)=0 \\ &= a_n^{2n-1}\prod_{i\neq j}(\alpha_i-\alpha_j) \\ &= a_n^{2n-1}(-1)^{n(n-1)/2}\prod_{i<j}(\alpha_i-\alpha_j)^2, && \text{by problem 4b on the homework} \end{aligned}$$

and so the alternate definition of the discriminant

$$D(f) = a_n^{2n-2}\prod_{i<j}(\alpha_i-\alpha_j)^2 = \frac{(-1)^{n(n-1)/2}}{a_n}R(f,f',t),$$

which explains the apparently unnecessary $\frac{(-1)^{n(n-1)/2}}{a_n}$ in defining the discriminant.

Of course, the discriminant $\prod_{i<j}(\alpha_i - \alpha_j)^2$ of a monic polynomial f with roots α_i, is the square of another polynomial in the roots. It turns out one of the two square roots is the *Vandermonde determinant*

$$(-1)^{n(n-1)/2}\prod_{i<j}(\alpha_i - \alpha_j) = \prod_{i<j}(\alpha_j - \alpha_i) = \det\begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \dots & \alpha_n^{n-1} \end{bmatrix}$$

so that the discriminant is the square of the Vandermonde determinant. Note that the Vandermonde determinant is not itself a symmetric polynomial in the α_i: interchanging α_i with α_j interchanges two rows of the matrix, multiplying the determinant by -1. This means that it can't possibly be a polynomial in the coefficients of f.

To prove that the Vandermonde determinant is equal to the product above, we can apply row and column operations to compute the determinant:

$$\begin{aligned}
\det\begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \dots & \alpha_n^{n-1} \end{bmatrix} &= \det\begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 0 & \alpha_2 - \alpha_1 & \alpha_2^2 - \alpha_1^2 & \dots & \alpha_2^{n-1} - \alpha_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_n - \alpha_1 & \alpha_n^2 - \alpha_1^2 & \dots & \alpha_n^{n-1} - \alpha_n^{n-1} \end{bmatrix} \\
&= \det\begin{bmatrix} \alpha_2 - \alpha_1 & \alpha_2^2 - \alpha_1^2 & \dots & \alpha_2^{n-1} - \alpha_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n - \alpha_1 & \alpha_n^2 - \alpha_1^2 & \dots & \alpha_n^{n-1} - \alpha_n^{n-1} \end{bmatrix} \\
&= \left(\prod_{i=2}^{n}(\alpha_i - \alpha_1)\right)\det\begin{bmatrix} 1 & \alpha_2 + \alpha_1 & \dots & \alpha_2^{n-2} + \dots + \alpha_1^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n + \alpha_1 & \dots & \alpha_n^{n-2} + \cdots \alpha_n^{n-2} \end{bmatrix} \\
&= \left(\prod_{i=2}^{n}(\alpha_i - \alpha_1)\right)\det\begin{bmatrix} 1 & \alpha_2 & \dots & \alpha_2^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \dots & \alpha_n^{n-2} \end{bmatrix}
\end{aligned}$$

where in the last step we are subtracting from each column multiples of earlier columns. The desired formula for the Vandermonde determinant then follows by induction on n.