1. Last week, you found that the following curves have only one singularity, at \( p = (0, 0) \), and calculated the Taylor series expansions at that point. Now, find the multiplicity of each curve at \( p \) and find the tangent cone \( TC_p(X) \). This should be a matter of interpreting the Taylor series calculations you have already made. Sketch the curves and draw in the tangent cones.

(a) \( f(x, y) = x^4 + y^4 - x^2 \)
(b) \( f(x, y) = x^6 + y^6 - xy \)
(c) \( f(x, y) = y^2 + x^4 + y^4 - x^3 \)
(d) \( f(x, y) = x^4 + y^4 - x^2y - xy^2 \)
(e) \( f(x, y) = x^3 + x^2 - y^2 \)
(f) \( f(x, y) = (x^2 + y^2)^2 - x^2 + y^2 \)

2. Use the same methods to find the singularities, the multiplicity at each singularity, and the tangent cones of the following curves. Since these are a bit more complicated, you will probably want to get a computer to do most of the calculations. Sketch a graph of the curve and its tangent cone near each singularity. Depending on what program you use, you may have to be careful of the behavior near singular points. Use your information from the tangent cone to interpret the behavior near singularities.

(a) \( f(x, y) = 2x^4 - 3x^2y + y^2 - 2y^3 + y^4 \)
(b) \( f(x, y) = 2y^2(x^2 + y^2) - 3y^2 - x^2 + 1 \)
(c) \( f(x, y) = 2y^2(x^2 + y^2) - 2y^2(x + y) - 2y^2 - x^2 + 2x + 2y \)
(d) \( f(x, y) = (x^2 + y^2)^3 - 4x^2y^2 \)

3. One can think of multiplicity as measuring how “bad” a singularity is. We already showed that for a nonsingular point on a curve, most lines intersect that point with multiplicity one.

(a) For the curve \( f(x, y) = x^3 - y^2 \), show that most lines through the origin meet the curve with multiplicity 2.
(b) For the curve \( g(x, y) = x^4 + 2xy^2 + y^4 \), show that most lines through the origin meet the curve with multiplicity \( \geq 3 \).

4. We’ve mentioned that we ought to be able to make a simple change coordinates so that, for example, a singular point is moved to the origin. The basic idea we were hinting at is that of affine equivalence. An \textit{affine change of coordinates} is a map of the form

\[
\phi \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}, \quad \text{where } \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0.
\]

We can think of this as basically just a change of variables (but one which is allowed to distort angles and distances). Two curves \( f(x, y) \) and \( g(x, y) \) are affine equivalent if they differ by an affine change of coordinates \( \phi \). That is, \( f(x, y) = g(\phi(x, y)) \).

Show that the curves \( f(x, y) = y^2 - x^3 - x^2 \) and \( g(x, y) = x^2 - 2xy - x - y + \frac{1}{4} - y^3 \) are affine equivalent.
5. Show that multiplicity is invariant under affine equivalence. That is, if \( \phi : C_1 \to C_2 \) is an affine equivalence, it maps a point with multiplicity \( m \) to a point with multiplicity \( m \).

6. This problem is a little different, and its connection to plane curves or algebraic geometry won’t be apparent for a while.

   (a) What natural numbers \( n \) are expressible in the form \( n = 2x + 3y \) where \( x \) and \( y \) are nonnegative integers? What if we allow \( x \) or \( y \) to be negative?

   (b) What natural numbers \( n \) are expressible in the form \( n = 4x + 6y \) where \( x \) and \( y \) are nonnegative integers? What if we allow \( x \) or \( y \) to be negative?

   (c) What natural numbers \( n \) are expressible in the form \( n = 5x + 8y \) where \( x \) and \( y \) are nonnegative integers? What if we allow \( x \) or \( y \) to be negative?