COMPLEX ANALYSIS

CHAPTER I
INTRODUCTION

Jan 7 $\mathbb{C}$

exp

HW 1 F

Jan 9 Re, Im, conjugate

cosh sinh cos sin

HW 2 M

Jan 11 roots of 1

logarithm

Möbius transformations

$\widehat{\mathbb{C}}$

HW 3 W

Jan 14 stereographic projection

Möbius again — circles

HW 4 W

$\boxed{\text{MATH 382 Lecture notes for M 1-7-13}}$

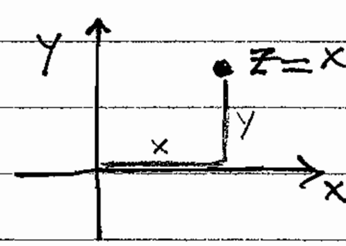
$\mathbb{C}$, the field of COMPLEX NUMBERS, is the set of all expressions of the form $x+iy$, where

- $x, y \in \mathbb{R}$
- i is a special symbol
- addition and multiplication with the usual rules, except
- $\boxed{i^2 = -1}$

The complex number 0 is simply $0+i0$. This is a field, since every complex number other than 0 has a multiplicative inverse:

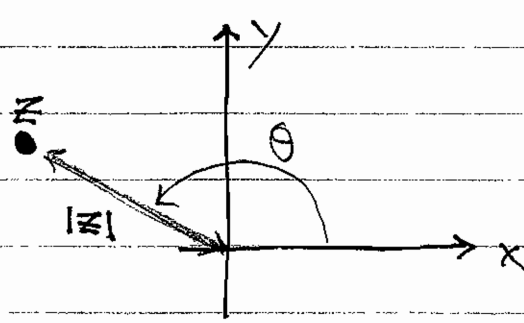
$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}.$$

CARTESIAN REPRESENTATION:



$z = x+iy$ is located at $(x,y) \in \mathbb{R}^2$

POLAR REPRESENTATION:



$|z| = \sqrt{x^2+y^2}$ = the modulus of z

The usual polar angle θ is called "the" argument of z: $\boxed{\arg z}$

All the usual care must be taken with $\arg z$, as there is not a unique determination of it. For instance

$$\arg(1+i) = \frac{\pi}{4} \text{ or } \frac{9\pi}{4} \text{ or } -\frac{7\pi}{4} \text{ or } \frac{201\pi}{4} \cdots.$$

$\boxed{\text{THE EXPONENTIAL FUNCTION}}$ is the function from $\mathbb{C}$ to $\mathbb{C}$ given by the power series

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots.$$

We shall soon discuss power series in detail, and we'll see immediately that the above series converges absolutely. We'll usually use the notation

$$e^z \text{ for } \exp(z).$$

PROPERTIES

- $\boxed{e^{z+w} = e^z e^w}$
- if $z \in \mathbb{R}$, e^z is the usual calculus function
- if $t \in \mathbb{R}$, then we have Euler's formula

$$\boxed{e^{it} = \cos t + i \sin t}.$$

We gave a sort of proof of the functional equation. If we ignore the convergence issues, the proof goes like this:

Proof #1 of $e^{z+w}=e^z e^w$

$$
\begin{aligned}
e^z e^w &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) \\
&= \left(\sum_{m=0}^{\infty} \frac{z^m}{m!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) && \text{change dummy} \\
&= \sum_{m,n=0}^{\infty} \frac{z^m}{m!}\frac{w^n}{n!} && \text{multiply the series} \\
&= \sum_{l=0}^{\infty} \sum_{m+n=l} \frac{z^m w^n}{m!\,n!} && \text{"diagonal" summation} \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{m=0}^{l} \frac{l!}{m!(l-m)!} z^m w^{l-m} && n=l-m \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{m=0}^{l} \binom{l}{m} z^m w^{l-m} && \longleftarrow \text{binomial coefficient} \\
&= \sum_{l=0}^{\infty} \frac{(z+w)^l}{l!} && \text{binomial formula} \\
&= e^{z+w}. && \text{definition}
\end{aligned}
$$

<u>Geometric description of complex multiplication</u>

The polar form helps us here. Suppose z and w are two nonzero complex numbers, and write

$$
\begin{aligned}
z &= |z|e^{i\theta}, && (\theta = \arg z) \\
w &= |w|e^{i\varphi} && (\varphi = \arg w).
\end{aligned}
$$

Then we have immediately that

$$zw = |z|\,|w|\,e^{i(\theta+\varphi)}.$$

We may thus conclude that the product zw has the polar coordinate data

$$|zw| = |z|\,|w|,$$
$$\arg(zw) = \arg z + \arg w.$$

Thus for a fixed $w \neq 0$, the operation of mapping z to zw

- multiplies the modulus by $|w|$,
- adds the quantity $\arg w$ to $\arg z$.

In other words, zw results from z by

- stretching by the factor $|w|$, and
- rotating by the angle $\arg w$.

HW 1F due at beginning of class Jan 11, 2013

Let a, b, c be three distinct complex numbers.
Prove that these numbers are the vertices of an equilateral triangle $\Longleftrightarrow$ $\boxed{a^2+b^2+c^2 = ab+bc+ca}$

(Suggestion: first show that translation of a,b,c does not change the equilateral triangle nature (clear) and also does not change the algebraic relation. Then show the same for multiplication of a,b,c by a fixed nonzero complex number.)

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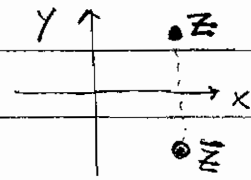
More $\mathbb{C}$ notation



Complex conjugate of $z = x+iy$ is $\bar{z} = x - iy$.

$\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$... so both real + imaginary parts of z are in $\mathbb{R}$.

- $z + \bar{z} = 2\,\mathrm{Re}(z)$
- $z - \bar{z} = 2i\,\mathrm{Im}(z)$
- $\overline{zw} = \bar{z}\,\bar{w}$
- $|z|^2 = z\bar{z}$

We can therefore observe that the important formula for $|zw|$ follows purely algebraically:

$$|zw|^2 = (zw)\overline{(zw)} = zw\bar{z}\bar{w} = z\bar{z}w\bar{w} = |z|^2|w|^2.$$

HW 1F SOLUTION

(This is certainly not the only possible solution.)

We are dealing with $a, b, c \in \mathbb{C}$, distinct and want to prove that they are vertices of an equilateral $\triangle \iff a^2+b^2+c^2 = ab+bc+ca$.

- TRANSLATION PRESERVES BOTH PROPERTIES, for

$$(a+d)^2+(b+d)^2+(c+d)^2 = a^2+b^2+c^2+\underline{2(ad+bd+cd)}+\underline{\underline{3d^2}},$$

$$(a+d)(b+d)+(b+d)(c+d)+(c+d)(a+d)$$
$$= ab+bc+ca+\underline{2(ad+bd+cd)}+\underline{\underline{3d^2}}.$$

- So we may move $\frac{a+b}{2}$ to 0. Thus WLOG we may assume $a = -b$.
- MULTIPLICATION ALSO PRESERVES BOTH PROPERTIES, so we may multiply by $\frac{1}{a}$ and thus may assume $a = 1$, $b = -1$.

The condition becomes $2 + c^2 = -1$, so $c^2 = -3$, so $c = \pm i\sqrt{3}$.

QED

More about exponential function In the power series for $\exp(z)$ split the terms into even and odd parts:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{z^n}{n!} + \sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} \frac{z^n}{n!}.$$

$$=: \cosh z + \sinh z.$$

In other words,

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

HYPERBOLIC COSINE HYPERBOLIC SINE

It's simple algebra to derive the corresponding addition formulas, just using $e^{z+w} = e^z e^w$. For instance,

$$\begin{aligned}
2\sinh(z+w) &= e^{z+w} - e^{-z-w} \\
&= e^z e^w - e^{-z} e^{-w} \\
&= (\cosh z + \sinh z)(\cosh w + \sinh w) \\
&\quad - (\cosh z - \sinh z)(\cosh w - \sinh w) \\
&\overset{\text{algebra}}{=} \cosh z \cosh w + \cosh z \sinh w + \sinh z \cosh w + \sinh z \sinh w \\
&\quad - \;\;'' \quad + \;\;'' \quad + \;\;'' \quad - \;\;'' \\
&= 2\sinh z \cosh w + 2\cosh z \sinh w.
\end{aligned}$$

Thus

- $\sinh(z+w) = \sinh z \cosh w + \cosh z \sinh w$.

Likewise,

- $\cosh(z+w) = \cosh z \cosh w + \sinh z \sinh w$.

Trigonometric functions By definition for all $z \in \mathbb{C}$ we have

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

$$\sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

There's a simple relation between the hyperbolic functions and the trigonometric ones:

$$\boxed{\begin{aligned} \cosh(iz) &= \cos z \\ \sinh(iz) &= i \sin z \end{aligned}}$$

Conversely,

$$\boxed{\begin{aligned} \cos(iz) &= \cosh z \\ \sin(iz) &= i \sinh z \end{aligned}}$$

The definitions of cos and sin can also be expressed this way:

$$\boxed{\begin{aligned} \cos z &= \frac{e^{iz}+e^{-iz}}{2}, \\ \sin z &= \frac{e^{iz}-e^{-iz}}{2i} \end{aligned}}$$

We also immediately derive

- $\sin(z+w) = \sin z \cos w + \cos z \sin w$,
- $\cos(z+w) = \cos z \cos w - \sin z \sin w$.

notice!

HW 2M due at beginning of class Jan 14, 2013

- Show that $|\sinh z|^2 = \sinh^2 x + \sin^2 y$.

Likewise,

- show that $|\cosh z|^2 = (?)^2 + (?)^2$.

More geometrical aspects of $\mathbb{C}$

We shall frequently need to deal with the modulus of a sum, and here's the easy algebra:

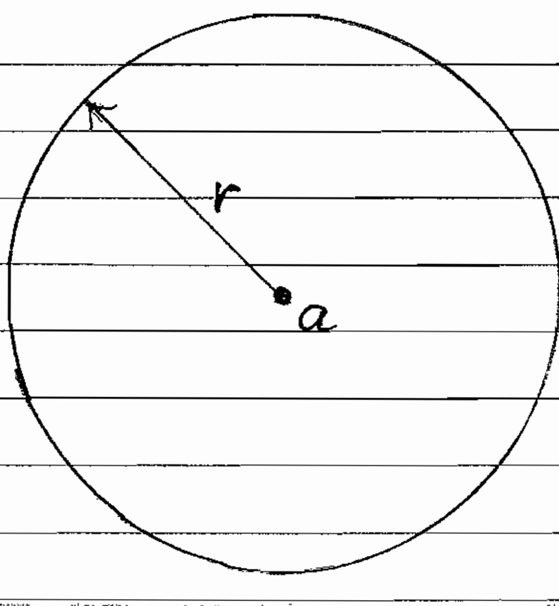
$$
\begin{aligned}
|z+w|^2 &= (z+w)\overline{(z+w)} \\
&= (z+w)(\bar{z}+\bar{w}) \\
&= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\
&= |z|^2 + 2\mathrm{Re}(z\bar{w}) + |w|^2.
\end{aligned}
$$

Displayed for reference, I'll call this the

$$\text{LAW OF COSINES: } \boxed{|z+w|^2 = |z|^2 + 2\mathrm{Re}(\bar{z}w) + |w|^2}$$

As an illustration let us write down the equation of a circle in $\mathbb{C}$. Suppose the circle has center $a \in \mathbb{C}$ and radius $r > 0$. Then z is on the circle $\iff$ $|z-a| = r$. That is, according to the above formula,

$$\boxed{|z|^2 - 2\mathrm{Re}(z\bar{a}) + |a|^2 = r^2.}$$



Dot product formula relating $\mathbb{R}^2$ and $\mathbb{C}$:

$$z \bullet w = \mathrm{Re}(z\bar{w})$$

$\mathbb{R}^2$ dot product

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HW 2M SOLUTION

Addition formula for sinh $\Rightarrow$

$$\sinh z = \sinh(x+iy) = \sinh x \cosh iy + \cosh x \sinh iy$$
$$= \sinh x \cos y + i \cosh x \sin y.$$

$\therefore$

$$|\sinh z|^2 = \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y$$
$$= \sinh^2 x\,(1-\sin^2 y) + \cosh^2 x \sin^2 y$$
$$= \sinh^2 x + \sin^2 y\,(-\sinh^2 x + \cosh^2 x)$$
$$= \sinh^2 x + \sin^2 y.$$

Same procedure of course takes care of $|\cosh z|$. Instead, we could use $\sinh(z + i\frac{\pi}{2}) = i\cosh z$ so that

$$|\cosh z|^2 = |\sinh(z + i\tfrac{\pi}{2})|^2$$
$$= |\sinh(x + i(y+\tfrac{\pi}{2}))|^2$$
$$= \sinh^2 x + \sin^2(y+\tfrac{\pi}{2})$$
$$= \sinh^2 x + \cos^2 y.$$

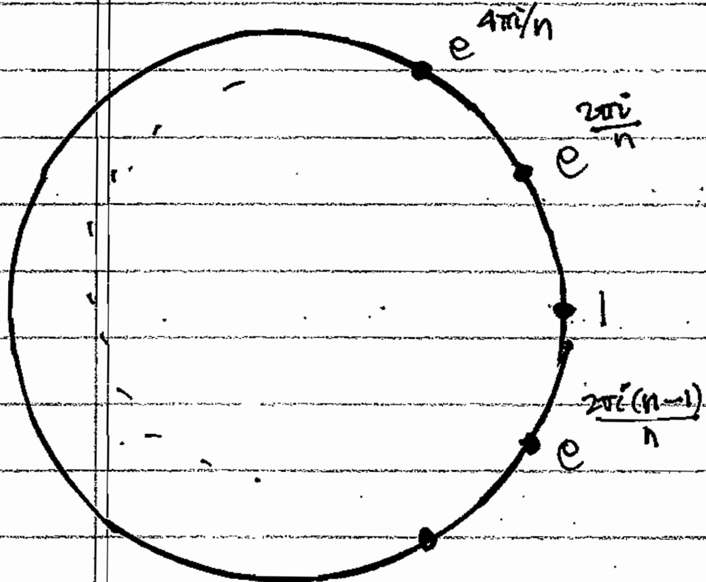
ROOTS OF UNITY

This is about the solutions of the equation $z^n = 1$, where n is a fixed positive integer.

We find n distinct roots, essentially by inspection:

$$z = e^{\frac{2\pi i k}{n}} \quad \text{for } k = 0, 1, \dots, n-1.$$

These are of course equally spaced points on the unit circle.



Simple considerations of basic polynomial algebra show that the polynomial $z^n - 1$ is exactly divisible by each factor $z - e^{2\pi i k/n}$. Therefore,

$$\boxed{z^n - 1 = \prod_{k=0}^{n-1} \left(z - e^{2\pi i k/n}\right),}$$

an identity for the polynomial $z^n - 1$.

COMPLEX LOGARITHM

This is about an inverse "function" for exp. In other words, we want to solve the equation $e^w = z$ for w. Of course, $z = 0$ is not allowed.

Quite easy: represent $w = u + iv$ in Cartesian form and $z = re^{i\theta}$ in polar form. Then we need

$$e^{u+iv} = re^{i\theta},$$
$$e^u e^{iv} = re^{i\theta},$$

this equation is true $\Leftrightarrow$ $e^u = r$ and $e^{iv} = e^{i\theta}$.

As $r > 0$, we have $\underline{u = \log r}$. Then $\underline{v = \theta + 2\pi \cdot \text{integer}}$.

As $\theta = \arg z$, we thus have the formula $w = \log r + i(\theta + 2\pi n)$, and we write

$$\boxed{\log z = \log|z| + i \arg z.}$$

COMPLEX LOG USUAL REAL LOG

Thus, $\log z$ and $\arg z$ share the same sort of ambiguity.

Properties

- $e^{\log z} = z$ (no ambiguity).
- $\log e^z = z$ (ambiguity of $2\pi n i$)
- $\log(zw) = \log z + \log w$ (with ambiguity)
- $\log(z^n) = n \log z$ for $n \in \mathbb{Z}$ (with ambiguity)

E.g.
$$\log(1+i\sqrt{3}) = \log 2 + i\frac{\pi}{3},$$
$$\log(-6) = \log 6 + i\pi,$$
$$\log(re^{i\theta}) = \log r + i\theta.$$

MÖBIUS TRANSFORMATIONS

This will be only a provisional definition, so that we'll simply become accustomed to the basic manipulations.

We want to deal with functions of the form

$$f(z) = \frac{az+b}{cz+d},$$

where a, b, c, d are complex constants. We do not want to include cases where f is constant, meaning that $az+b$ is proportional to $cz+d$. I.e., meaning that the vectors (a, b) and (c, d) in $\mathbb{C}^2$ are linearly dependent. A convenient way to place this restriction is to require $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$. This we shall always require.

Easy calculation: if $g(z) = \dfrac{a'z+b'}{c'z+d'}$, then the composition $f \circ g$ $\left((f\circ g)(z) = f(g(z))\right)$ corresponds to the matrix product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

If
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \quad (\text{with } \lambda \neq 0),$$
then these two matrices give the same transformation.

$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the <u>extended</u> complex plane, and we then also define

$$f(-d/c) = \infty,$$
$$f(\infty) = a/c.$$

(We'll have <u>much</u> more to say about these formulas a little later.)

The functions we have defined this way are called <u>Möbius transformations</u>. Each of them gives a <u>bijection</u> of $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$. And each of them has a unique inverse:

$$f(z) = \frac{az+b}{cz+d} \Rightarrow f^{-1}(z) = \frac{dz-b}{-cz+a}.$$

HW 3W due Jan 16, 2013

Let C be the circle in $\mathbb{C}$ with center $a \in \mathbb{C}$, radius $r > 0$.
(From Jan 9 we know $z \in C \Leftrightarrow$

$$|z|^2 - 2\,\mathrm{Re}(z\bar{a}) + (|a|^2 - r^2_\bullet)$$

We want to investigate the outcome of forming $\frac{1}{z}$ for all $z \in C$.

1. If $0 \notin C$, define

$$D = \left\{ \frac{1}{z} \,\middle|\, z \in C \right\}.$$

Prove that D is also a circle, and calculate its <u>center</u> and <u>radius</u>:

center $=$?

radius $=$?

2. If $0 \in C$, then instead define

$$D = \left\{ \frac{1}{z} \,\middle|\, z \in C,\ z \neq 0 \right\}.$$

What geometric set is D? Prove it.

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HW 3W SOLUTION

Given circle $C \subset \mathbb{C}$

it's described as the set of all $z \in \mathbb{C}$ s.t.

$$|z|^2 - 2\mathrm{Re}(z\bar{a}) + |a|^2 - r^2 = 0.$$

We want to investigate all the complex numbers $\frac{1}{z}$ for $z \in \mathbb{C}$. Let's define $w = \frac{1}{z}$. Then w is determined by the equation

$$\left|\frac{1}{w}\right|^2 - 2\mathrm{Re}\left(\frac{\bar{a}}{w}\right) + |a|^2 - r^2 = 0;$$

multiply through by $|w|^2 = w\bar{w}$:

$$1 - 2\mathrm{Re}(\bar{w}\,\bar{a}) + (|a|^2 - r^2)|w|^2 = 0.$$

1. $\underline{0 \notin C}$ This means that $|a|^2 - r^2 \neq 0$. So the equation becomes

$$|w|^2 - 2\mathrm{Re}\left(w\frac{a}{|a|^2 - r^2}\right) + \frac{1}{|a|^2 - r^2} = 0.$$

Aha! This describes a $\underline{\text{circle}}$ with center

$$a' = \frac{\bar{a}}{|a|^2 - r^2}$$

and radius r', where

$$|a'|^2 - r'^2 = \frac{1}{|a|^2 - r^2}.$$

Thus

$$r'^2 = |a'|^2 - \frac{1}{|a|^2 - r^2}$$

$$= \frac{|a|^2}{(|a|^2 - r^2)^2} - \frac{1}{|a|^2 - r^2}$$

$$= \frac{|a|^2 - (|a|^2 - r^2)}{(|a|^2 - r^2)^2}$$

$$= \frac{r^2}{(|a|^2 - r^2)^2}.$$

Therefore,

$$\boxed{\text{new center} = \frac{\bar{a}}{|a|^2-r^2},}$$

$$\boxed{\text{new radius} = \frac{r}{\left||a|^2-r^2\right|}.}$$

2. $\underline{0 \in C}$ In this case $|a| = r$, so the equation for w becomes

$$1 - 2\,\mathrm{Re}(wa) = 0.$$

Using Cartesian coordinates $a = b + ic$, $w = u + iv$, this equation is

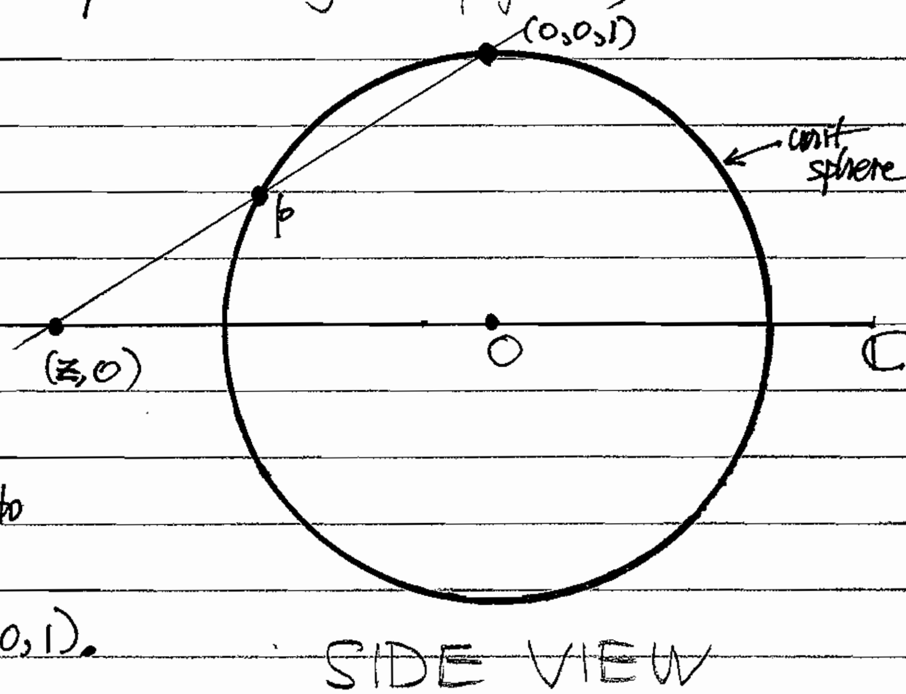
$$bu - cv = \frac{1}{2},$$

so D is a **straight line.**

<u>More about the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$</u>

This enjoys a beautiful geometric depiction as the <u>unit sphere in $\mathbb{R}^3$</u>, by means of <u>stereographic projection</u>, which we now describe. There are several useful ways of defining this projection, but I choose the following:

Let $\mathbb{R}^3$ be given Cartesian coordinates (x, y, t), where $z = x + iy$.



Project unit sphere onto $\mathbb{C}$ $(t=0)$ from the north pole $(0,0,1)$.

Straight lines through the north pole which are not horizontal intersect the plane $t=0$ and the unit sphere and set up a bijection between $\mathbb{C}$ and the unit sphere minus $(0,0,1)$, as shown in the figure.

When $|z| \to \infty$ the projection $p \to (0,0,1)$. Thus by decreeing that the north pole corresponds to some point, we are led to adjoining ∞ to $\widehat{\mathbb{C}}$.

Thus $\widehat{\mathbb{C}}$ is "equivalent" to the unit sphere in $\mathbb{R}^3$, so $\widehat{\mathbb{C}}$ is often called the **Riemann sphere**.

More about Möbius transformations

- Baby case: given 3 distinct complex numbers a, b, c, it's an easy matter to find a Möbius f such that

$$\begin{cases} f(a)=0 \\ f(b)=\infty \\ f(c)=1. \end{cases}$$

In fact, f is uniquely determined, and we must have

$$(*) \quad f(z) = \frac{z-a}{z-b}\,\frac{c-b}{c-a}.$$

- Embellishment: we can even allow a or b or c to be ∞, and again there's a unique Möbius f. Here are the results:

$f(\infty)=0$, $f(b)=\infty$, $f(c)=1$: $f(z)=\frac{c-b}{z-b}$	$f(a)=0$, $f(\infty)=\infty$, $f(c)=1$; $f(z)=\frac{z-a}{c-a}$	$f(a)=0$, $f(b)=\infty$, $f(\infty)=1$; $f(z)=\frac{z-a}{z-b}$

(REMARK: each case results from $(*)$ by replacing a, b, c by ∞ formally.)

- General case: given 3 distinct points $a, b, c \in \widehat{\mathbb{C}}$
  and also 3 distinct points $a', b', c' \in \widehat{\mathbb{C}}$,
  then there is a unique Möbius f such that

$$\begin{cases} f(a) = a' \\ f(b) = b' \\ f(c) = c'. \end{cases}$$

<u>Proof</u> Use the previous case twice:

$$\begin{tikzcd} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \arrow[r, dashed, "f"] \arrow[d, "g"] & \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \arrow[d, "h"] \\ \begin{pmatrix} 0 \\ \infty \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ \infty \\ 1 \end{pmatrix} \end{tikzcd}$$

Then $f = h^{-1} \circ g$.

QED

<u>Möbius transformations and circles</u>

According to HW3 the image of a circle under the action of $z \mapsto \frac{1}{z}$ is another circle (or straight line). <u>The same is true if instead of $\frac{1}{z}$ we use any Möbius transformation.</u> Let

$$f(z) = \frac{az+b}{cz+d}.$$

<u>Case 1</u> <u>$c=0$</u> Then we may as well write $f(z) = az+b$.
This transformation involves multiplication by $|a|$, rotation by $\arg a$, and translation by b. Thus circles are preserved by f.

<u>Case 2</u> <u>$c \neq 0$</u> Then we may as well write $f(z) = \frac{az+b}{z+d}$, where $ad - b \neq 0$. But then

$$f(z) = \frac{a(z+d)}{z+d} + \frac{b-ad}{z+d} = a + \frac{b-ad}{z+d},$$

so f is given by translation, then reciprocation, then multiplication, then translation. All operations preserve "circles" if we include straight lines.

HW 4W due Jan 23, 2013

Start from the result we obtained on Jan 11: if $n \geq 2$ is an integer, then

$$z^n - 1 = \prod_{k=0}^{n-1} \left(z - e^{2\pi i k/n}\right).$$

1. Prove that for any $z, w \in \mathbb{C}$

$$z^n - w^n = \prod_{k=0}^{n-1} \left(z - w e^{2\pi i k/n}\right).$$

2. Prove that

$$z^n - w^n = \prod_{k=0}^{n-1} \left(z - w\, e^{-2\pi i k/n}\right).$$

3. Prove that

$$z^n - w^n = (-i)^{n-1} \prod_{k=0}^{n-1} \left(e^{\pi i k/n} z - e^{-\pi i k/n} w\right).$$

4. Replace z by e^{iz} and w by e^{-iz} and show that

$$\boxed{\sin nz = 2^{n-1} \prod_{k=0}^{n-1} \sin\left(z + \frac{\pi k}{n}\right).}$$

5. Show that

$$\boxed{\prod_{k=1}^{n-1} \sin \frac{\pi k}{n} = \frac{n}{2^{n-1}}.}$$

6. Prove that $\cos z = \cos w \iff \left\{ \begin{array}{c} z - w = 2k\pi \\ \text{or} \\ z + w = 2k\pi \end{array} \right\}$ for some $k \in \mathbb{Z}$.

7. Prove that $\sin z = \sin w \iff \left\{ \begin{array}{c} z - w = 2k\pi \\ \text{or} \\ z + w = \,? \end{array} \right\}$ for some $k \in \mathbb{Z}$.