COMPLEX ANALYSIS

CHAPTER II
DIFFERENTIATION

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EXAM 1

MATH 382 Lecture notes for 1-16-13

DIFFERENTIATION

Today we began a thrilling introduction to complex analysis. It all starts with a seemingly innocent and reasonable definition of derivative, using complex numbers instead of real numbers. But we shall learn very soon what an enormous step this really is!

DEFINITION Let f be a complex valued function defined on some neighborhood of a point $z \in \mathbb{C}$. We say that f is **complex-differentiable** at z if

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}$$

CRUCIAL!

exists.

In case this limit exists, it's called the **complex derivative** of f at z, and is denoted either

$$f'(z) \quad \text{or} \quad \frac{df}{dz}.$$

This truly seems naive, as it's completely similar to the beginning definition in calculus. But we shall see that the properties of f which follow from this definition are <u>astonishing</u>!

What makes this all so powerful is that in the difference quotient the denominator $h \in \mathbb{C}$ must be allowed simply to tend to 0, no restrictions on "how" or particular directions: merely $|h| \to 0$.

Jan 16(2)

BASIC PROPERTIES

- $f'(z)$ exists $\Longrightarrow$ f is continuous at z.

  For if $\dfrac{f(z+h)-f(z)}{h}$ has a limit, then since $h \to 0$, the numerator must also have limit 0, so that
  $$\lim_{h\to 0} f(z+h) = f(z).$$

- f + g differentiable $\Longrightarrow$ $f+g$ is too, and $(f+g)' = f' + g'$.
- PRODUCT RULE: also fg is differentiable, and
  $$(fg)' = fg' + f'g.$$

  Proof:
  $$\frac{f(z+h)g(z+h) - f(z)g(z)}{h}$$
  $$= \underbrace{f(z+h)}_{\substack{\downarrow \\ f(z) \\ \text{by continuity}}} \underbrace{\frac{g(z+h)-g(z)}{h}}_{\substack{\downarrow \\ g'(z)}} + g(z) \underbrace{\frac{f(z+h)-f(z)}{h}}_{\substack{\downarrow \\ f'(z)}}$$

- $\dfrac{dz}{dz} = 1$ and then we prove by induction that for $n = 1, 2, 3, \ldots$
  $$\frac{dz^n}{dz} = nz^{n-1}$$

- QUOTIENT RULE $\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$ provided that $g \neq 0$.

- CHAIN RULE: If g is differentiable at z and f is differentiable at $g(z)$, then $f \circ g$ is differentiable at z, and
  $$(f \circ g)'(z) = f'(g(z))\, g'(z).$$

All these properties are proved just as in "real" calculus, so I've not bothered to write out detailed proofs of them all.

EXAMPLES

- <u>Möbius transformations</u> — directly from the quotient rule

$$\left(\frac{az+b}{cz+d}\right)' = \frac{ad-bc}{(cz+d)^2} \leftarrow \text{notice the determinant!}$$

- <u>Exponential function</u>

First for $h \to 0$ we have

$$\frac{e^h-1}{h} = \sum_{n=1}^{\infty} \frac{h^{n-1}}{n!} = 1 + \frac{h}{2} + \frac{h^2}{6} + \dots$$

has limit 1 as $h \to 0$. Thus

$$\frac{e^{z+h}-e^z}{h} = e^z \, \frac{e^h-1}{h} \to e^z.$$

Conclusion:

$$\frac{de^z}{dz} = e^z.$$

- <u>Trigonometric + hyperbolic functions</u> (follow immediately from exp)

$$\frac{d\sin z}{dz} = \cos z, \quad \frac{d\cos z}{dz} = -\sin z,$$

$$\frac{d\sinh z}{dz} = \cosh z, \quad \frac{d\cosh z}{dz} = \sinh z.$$

$$\boxed{\frac{d}{dz} \text{ and } \frac{\partial}{\partial x} \text{ and } \frac{\partial}{\partial y}}$$

By an <u>audacious</u> — but <u>useful</u> — abuse of notation we write

$$f(z) = f(x+iy) \overset{!}{=} f(x,y).$$

This sets up a correspondence between a function <u>defined on $\mathbb{C}$</u> and a function <u>defined on $\mathbb{R}^2$</u>, but we use the same name for those functions!

Now suppose $f'(z)$ exists. In the definition we then restrict h to be real ... the limit still exists, of course, and we compute

$$f'(z) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(z+h)-f(z)}{h} = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(x+h,y)-f(x,y)}{h}$$

$$= \frac{\partial f}{\partial x}(x,y).$$

Likewise, let $h = it$:

$$f'(z) = \lim_{\substack{t\to 0\\ t\in\mathbb{R}}} \frac{f(z+it)-f(z)}{it} = \lim_{\substack{t\to 0\\ t\in\mathbb{R}}} \frac{f(x,y+t)-f(x,y)}{it}$$

$$= \frac{1}{i}\frac{\partial f}{\partial y}(x,y).$$

We thus conclude that

$$f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

This second equality is a famous relationship, called

THE CAUCHY–RIEMANN EQUATION:

$$\boxed{\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}}$$

WARNING — everyone else calls this the Cauchy–Riemann equations, after expressing f in terms of its real and imaginary parts as $f = u + iv$. Then we indeed get 2 equations:

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}}$$

In a very precise sense, the converse is also valid, as we now discuss.

We suppose that f is differentiable at (x,y) <u>in the multivariable calculus sense</u>. This means that not only do the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (x,y), but also they provide the coefficients for a <u>good</u> <u>linear</u> <u>approximation</u> to $f(z+h)-f(z)$ for small $|h|$:

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}^2}} \frac{f(z+h)-f(z)-\frac{\partial f}{\partial x}(z)h_1-\frac{\partial f}{\partial y}(z)h_2}{|h|}=0.$$

(Remember: $z=x+iy$ is fixed.) We've denoted $h=h_1+ih_2$.

That definition actually extends to $\mathbb{R}^n$ just as well as $\mathbb{R}^2$. But in $\mathbb{R}^2$ we have an advantage in that we can replace the denominator $|h|$ by the complex number h without disturbing the fact that the limit is 0:

$$\lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{f(z+h)-f(z)-\frac{\partial f}{\partial x}(z)h_1-\frac{\partial f}{\partial y}(z)h_2}{h}=0.$$

Now <u>assume that the Cauchy-Riemann equation is satisfied</u>. Then we may replace $\frac{\partial f}{\partial y}$ by $i\frac{\partial f}{\partial x}$ and thus conclude that

$$\lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{f(z+h)-f(z)-\frac{\partial f}{\partial x}(z)\left(h_1+ih_2\right)}{h}=0.$$

i.e.,
$$\lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{f(z+h)-f(z)}{h}=\frac{\partial f}{\partial x}(z).$$

Therefore, we conclude that $f'(z)$ exists, so f is differentiable in the complex sense!

MATH 382 Lecture notes for 1-18-13

Cauchy–Riemann equation in polar coordinates

We employ the usual polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad (z = re^{i\theta}) \quad (r > 0 \text{ of course})$$

and then abuse notation by writing $f = f(x,y)$ as

$$f = f(r\cos\theta, r\sin\theta)$$

and then computing the r and θ partial derivatives of this composite function and designating them as $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ (terrible!).
Then the chain rule gives

$$\begin{cases} \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta, \\ \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta). \end{cases}$$

Now suppose f satisfies the Cauchy–Riemann equation and substitute $\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}$:

$$\begin{cases} \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}(\cos\theta + i\sin\theta), \\ \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta + ir\cos\theta). \end{cases}$$

Thus,

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}e^{i\theta},$$
$$\frac{1}{ir}\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}e^{i\theta}.$$

We conclude that

$$\boxed{\frac{\partial f}{\partial r} = \frac{1}{ir}\frac{\partial f}{\partial \theta}.}$$

polar coordinate form of the Cauchy–Riemann equation

Our calculations show that since $\frac{\partial f}{\partial x} = f'$,

$$f'(z) = e^{-i\theta}\frac{\partial f}{\partial r} = \frac{1}{ir}e^{-i\theta}\frac{\partial f}{\partial \theta}.$$

EXERCISE Prove that $\frac{\partial f}{\partial r} = \frac{1}{ir}\frac{\partial f}{\partial \theta}$ implies the original Cauchy–Riemann equation.

Complex logarithm

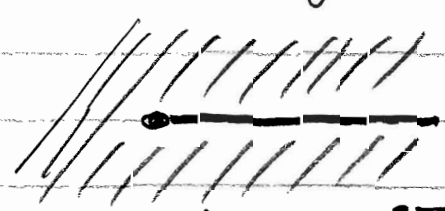
We've derived the defining equation

$$\log z = \log|z| + i \arg z.$$

In terms of polar coordinates,

$$\log z = \log r + i\theta.$$

We pause to discuss an easy but crucial idea. When we are faced with the necessity of using log or arg, we almost always work in a certain region of $\mathbb{C} \sim \{0\}$ in which it's possible to define $\arg z$ <u>in a continuous manner</u>. A typical situation might be the following: <u>exclude the nonnegative real axis</u> and define $\arg z$ so that $0 < \arg z < 2\pi$:



then we would have e.g.

$$\log(-1) = \pi i, \quad \log(ei) = 1 + i\frac{\pi}{2}, \text{ etc.}$$

In such a situation $\log z$ is also a well defined function of z, and the polar form of the Cauchy–Riemann equation applies immediately:

$$\frac{\partial}{\partial r}\log z = \frac{\partial}{\partial r}(\log r + i\theta) = \frac{1}{r},$$

$$\frac{\partial}{\partial \theta}\log z = \frac{\partial}{\partial \theta}(\log r + i\theta) = i;$$

thus

$$\frac{\partial}{\partial r}\log z = \frac{1}{ir}\frac{\partial}{\partial \theta}\log z.$$

Thus $\log z$ has a complex derivative, which equals $e^{-i\theta}\frac{1}{r} = 1/re^{i\theta} = 1/z$. We've thus obtained the expected formula

$$\frac{d \log z}{dz} = \frac{1}{z}.$$

(Be sure to notice that although $\log z$ is ambiguous, the ambiguity is of the form of an <u>additive constant</u> $2\pi i n$, so d/dz annihilates that constant.)

Now an extremely important definition will be given and discussed:

DEFINITION Let $D \subset \mathbb{C}$ be an open set, and assume that $D \xrightarrow{f} \mathbb{C}$ is a function which is of class C^1. That is, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are defined at each point of D and are themselves continuous functions on D.

Suppose also that $f'(z)$ exists at every point $z \in D$.

Then we say that f is a **<u>holomorphic</u> function** on D.

So of course we have at our disposal quite an array of holomorphic functions:

- exp
  and sinh, cosh, sin, cos
- all Möbius transformations
- log
- all <u>polynomials</u> in z: $f(z) = a_0 + a_1 z + \cdots + a_n z^n$
- all <u>rational</u> functions in z: $\dfrac{\text{polynomial}}{\text{polynomial}}$

We had quite a long and interesting discussion about this concept. Here are major points.

1. We don't actually need to <u>say</u> that D is an open set! The very existence of $f'(z)$ is that

$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}$$

and this requires $f(z+h)$ to be defined for all sufficiently small $|h|$, and thus that f is defined in some neighborhood of z.

2. The assumption that $f \in C^1$ can be dispensed with, as a fairly profound theorem implies that it follows from just the assumption that $f'(z)$ <u>exists</u> for every $z \in D$.

3. "**Holomorphic**" is not a word you'll see in most basic books on complex analysis. Usually those books use the word "**analytic**."

However, I want us to use "**analytic**" function to refer to a function which in a neighborhood of each z_0 in its domain can be represented as a <u>power series</u>

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with a positive radius of convergence.

- It's pretty easy to prove (and we shall do so) that every analytic function is holomorphic.
- A much more profound theorem will also be proved — that <u>every holomorphic function is analytic</u>.

Jan 18(5)

HW 4W SOLUTIONS

1. Start with the product decomposition of $z^n - 1$. Replace z by $\frac{z}{w}$, assuming $w \neq 0$:

$$\frac{z^n}{w^n} - 1 = \prod_{k=0}^{n-1} \left(\frac{z}{w} - e^{2\pi i k/n} \right);$$

multiply both sides by w^n:

$$z^n - w^n = \prod_{k=0}^{n-1} \left(z - w e^{2\pi i k/n} \right).$$

Finally, if $w = 0$, then the equation is trivial.

2. Express the nth roots of unity in clockwise order to get the result. Or conjugate the equation from 1 to get

$$\bar{z}^n - \bar{w}^n = \prod_{k=0}^{n-1} \left(\bar{z} - \bar{w} e^{-2\pi i k/n} \right)$$

and then replace z + w by their conjugates.

3. We have from 2

$$z^n - w^n = \prod_{k=0}^{n-1} e^{-\pi i k/n} \left(e^{\pi i k/n} z - e^{-\pi i k/n} w \right)$$

$$= \underbrace{\prod_{k=0}^{n-1} e^{-\pi i k/n}} \cdot \prod_{k=0}^{n-1} \left(e^{\pi i k/n} z - e^{-\pi i k/n} w \right).$$

This product equals $e^{-\sum_{k=0}^{n-1} \pi i k/n} = e^{-\frac{\pi i}{n} \cdot n \frac{(n-1)}{2}}$

$$= e^{-\frac{\pi i}{2}(n-1)} = \left(e^{-\pi i/2} \right)^{n-1} = (-i)^{n-1}.$$

4. Replace z by e^{iz} and w by e^{-iz}:

$$e^{inz} - e^{-inz} = (-i)^{n-1} \prod_{k=0}^{n-1} \left(e^{i(z + \frac{\pi k}{n})} - e^{-i(z + \frac{\pi k}{n})} \right).$$

That is, $$2i \sin nz = (-i)^{n-1} \prod_{k=0}^{n-1} 2i \sin\left(z + \frac{\pi k}{n}\right);$$

$$\sin nz = \underbrace{(-i)^{n-1}(2i)^{n-1}}_{(-i2i)^{n-1} = 2^{n-1}} \prod_{k=0}^{n-1} \sin\left(z + \frac{\pi k}{n}\right).$$

5. Therefore $$\frac{\sin nz}{\sin z} = 2^{n-1} \prod_{k=1}^{n} \sin\left(z + \frac{\pi k}{n}\right).$$

Now the right side is continuous even at $z = 0$. So we conclude that

$$\lim_{z \to 0} \frac{\sin nz}{\sin z} = 2^{n-1} \prod_{k=1}^{n} \sin \frac{\pi k}{n}.$$

There are several ways to evaluate this limit directly... including l'Hôpital's rule if you are careful. (More about this later.) Another is to use the power series for $\sin z$ to show that

$$\lim_{z \to 0} \frac{\sin z}{z} = 1.$$

Therefore

$$\frac{\sin nz}{\sin z} = \frac{\sin nz}{nz} \cdot \frac{z}{\sin z} \cdot n \longrightarrow 1 \cdot 1 \cdot n = n.$$

6. $$\cos z = \cos w \iff \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iw} + e^{-iw}}{2}$$

$$\iff e^{iz} - e^{iw} = e^{-iw} - e^{-iz}$$

$$\iff e^{iz} - e^{iw} = e^{-i(w+z)}\left(e^{iz} - e^{iw}\right)$$

$$\iff \left(e^{iz} - e^{iw}\right)\left(1 - e^{-i(w+z)}\right) = 0$$

$$\iff e^{iw}\left(e^{i(z-w)} - 1\right)\left(e^{-i(w+z)} - 1\right) = 0$$

$$\iff e^{i(z-w)} - 1 = 0 \text{ or } e^{-i(w+z)} - 1 = 0$$

$$\iff z - w = 2k\pi \text{ or } w + z = 2k\pi.$$

7. $\sin z = \cos\left(\frac{\pi}{2} - z\right)$, so $$\sin z = \sin w \iff \left(\frac{\pi}{2} - z\right) - \left(\frac{\pi}{2} - w\right) = 2k\pi$$
$$\overset{\text{or}}{+} \left(\frac{\pi}{2} - z\right) + \left(\frac{\pi}{2} - w\right) = 2k\pi$$

$$\iff z - w = 2j\pi \text{ or } \underline{z + w = (2j-1)\pi}.$$

Jan 18 (7)

HW 5M due January 28, 2013

Let D be the open half plane

$$D = \{ z \in \mathbb{C} \mid Re(z) > 0 \}.$$

Let f be the function defined on D by $f(z) = z^2$. Of course, f is holomorphic.

1. Prove that f is a bijection of D onto a set $D' \subset \mathbb{C}$.

2. What is D'?

3. The inverse function f^{-1} maps D' onto D. We'll actually prove a general theorem asserting that <u>inverses of holomorphic functions are always holomorphic</u>. But in this problem I want you to prove directly that f^{-1} is holomorphic.

4. For every real number $0 < a < \infty$ let L_a be the straight line

$$L_a = \{ z \in \mathbb{C} \mid Re(z) = a \}.$$

Prove that the images $f(L_a)$ are parabolas.

5. Prove that the *focus* of each parabola $f(L_a)$ is the origin.

6. For each real number b let M_b be the ray

$$M_b = \{ z \in \mathbb{C} \mid Im(z) = b \}.$$

Since f is conformal, the sets $f(M_b)$ and the parabolas $f(L_a)$ are orthogonal to one another.

Describe the sets $f(M_b)$.

MATH 382 Lecture notes for 1-23-13

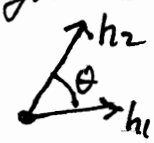
CONFORMAL TRANSFORMATIONS

Roughly speaking, the adjective <u>conformal</u> refers to the <u>preservation of angles</u>. More specifically, consider a situation in which a function F from one type of region to another is differentiable in the vector calculus sense. And consider a point p and its image $F(p)$. Calculus then enables us to move tangent vectors at p to tangent vectors at $F(p)$... some sort of notation like this is frequently used:

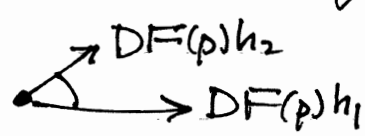
$$h \text{ (tangent vector at } p) \longrightarrow DF(p)h.$$

Here $DF(p)$ is often the <u>Jacobian matrix</u> of F at p and the symbol $DF(p)h$ refers to multiplication of a matrix and a vector.

Then if h_1 and h_2 are tangent vectors at p, they have a certain angle θ between them:



we are interested in the angle between the images under F of those tangent vectors:



If this angle is also θ and this happens at every p and for all tangent vectors, we say that F is a <u>conformal</u> transformation. Tersely,

$$\boxed{\text{conformal means angle-preserving.}}$$

Examples: <u>Mercator</u> projections of the earth;
<u>stereographic</u> projections.

Now we specify this for <u>holomorphic functions</u>. So assume that f is holomorphic and that at a fixed point z we know that $f'(z) \neq 0$. Let the polar form of this number be

$$f'(z) = Ae^{i\alpha} \quad (\text{where } A > 0, \alpha \in \mathbb{R}).$$

By definition

$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}.$$

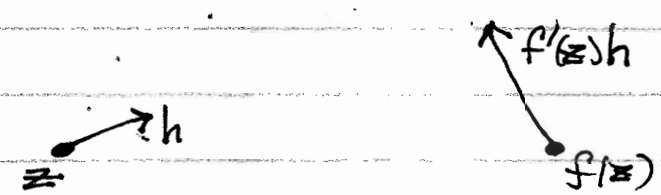
Rewrite this relationship as

$$f(z+h) = f(z) + f'(z)h \quad \text{approximately.}$$

This means that f transforms a <u>tangent vector</u> h at z to the vector at $f(z)$ given by

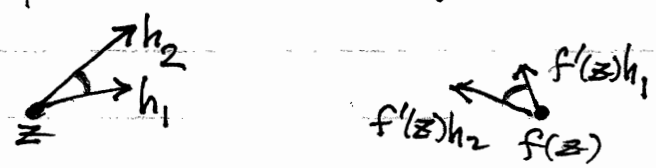
$$f'(z)h.$$

In other words, <u>directions</u> h at z are transformed to <u>directions</u> $f'(z)h$ at $f(z)$:



This action does two things to h: (1) multiplies its modulus by A and (2) rotates it by the angle α.

We conclude immediately that f preserves angles:

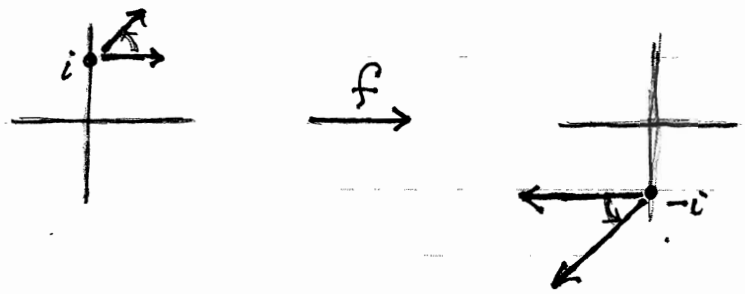


The moduli of all the infinitesimal vectors at z are multiplied by the same positive number A.

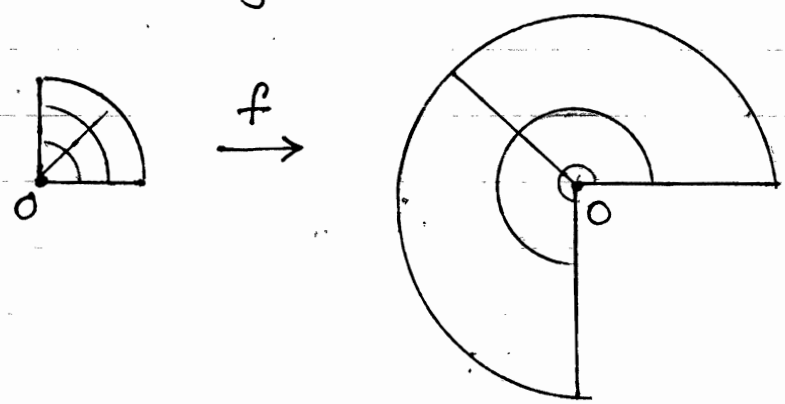
SUMMARY: Every holomorphic function f is conformal at every z with $f'(z) \neq 0$. Infinitesimal vectors at z are magnified by the positive number $|f'(z)|$.

EXAMPLE: $f(z) = z^3$.

$z = i \longrightarrow f'(i) = -3$
$f(i) = -i$



But notice that $f'(0) = 0$ and f does not preserve angles at 0 — instead, it multiplies them by 3.



(COMPLEX) POWER SERIES

1. Infinite series of complex numbers

We shall need to discuss $\sum_{n=0}^{\infty} a_n$, where $a_n \in \mathbb{C}$.

Convergence of such a series is no mystery at all. We form the sequence of partial sums

$$s_N = a_0 + \cdots + a_N,$$

and just demand that

$$\lim_{N\to\infty} s_N = L \quad \text{exist.}$$

Then we say

$$\sum_{n=0}^{\infty} a_n = L \quad \text{is convergent.}$$

Equivalently, we could reduce everything to two real series, require that they converge, and then

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \mathrm{Re}(a_n) + i \sum_{n=0}^{\infty} \mathrm{Im}(a_n).$$

Necessarily, if a series converges, then $\lim_{n\to\infty} a_n = 0$. (For

$$a_N = s_N - s_{N-1} \to L - L = 0.)$$

Converse is of course false: the "harmonic series" $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

Absolute convergence is what we'll usually see. We say that $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges. Then there's an important

Theorem If a series converges absolutely, then it converges.

(The proof relies on the completeness of $\mathbb{R}$.)

2. <u>Most important example of a power series</u> — the **GEOMETRIC SERIES**

$$\sum_{n=0}^{\infty} z^n, \text{ where } z \in S.$$

By our necessity condition, if this series converges, then $z^n \to 0$. That is, $|z|^n = |z^n| \to 0$. That is, $|z| < 1$.

Conversely, suppose $|z| < 1$. Then

$$s_N = 1 + z + \cdots + z^N = \frac{1 - z^{N+1}}{1 - z} \quad (z \neq 1 \text{ of course})$$
$$= \frac{1}{1-z} - \frac{z^{N+1}}{1-z}.$$

Now simply note that $\left| -\frac{z^{N+1}}{1-z} \right| = \frac{|z|^{N+1}}{|1-z|} \to 0$ because $|z| < 1$.

SUMMARY:

$$\sum_{n=0}^{\infty} z^n \text{ converges} \iff |z| < 1.$$

And then it converges absolutely, and

$$\boxed{\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.}$$

3. <u>Definition</u> A <u>power series</u> <u>centered</u> <u>at</u> z_0 is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the <u>coefficients</u> a_n are complex numbers.

☆ Usually in developing the properties of such series we'll work with the center $z_0 = 0$.

Simple warning: the first term in this series is not really

$$a_0 (z - z_0)^0,$$

it is actually a lazy way of writing the constant a_0. A more legitimate expression would be

$$a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n \quad \text{... more ever bottom.}$$

THEOREM (easy but crucial!) If a power series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges when $z = z_1$, and if $|z_2| < |z_1|$, then it converges absolutely when $z = z_2$.

(easy) PROOF $\sum_0^{\infty} a_n z_1^n$ converges $\Rightarrow \lim_{n\to\infty} a_n z_1^n = 0$

$$\Rightarrow |a_n z_1^n| \leq \text{a constant } C \text{ for all } n \geq 0.$$

Therefore,

$$|a_n z_2^n| \leq C |z_1|^{-n} |z_2|^n = C \left(\frac{|z_2|}{|z_1|}\right)^n.$$

Since $\frac{|z_2|}{|z_1|} < 1$, the geometric series $\sum_{n=0}^{\infty} \left(\frac{|z_2|}{|z_1|}\right)^n$. Therefore,

$$\sum_{n=0}^{\infty} |a_n z_2^n| \text{ converges.}$$

That is,

$$\sum_{n=0}^{\infty} a_n z_2^n \text{ converges absolutely.}$$

QED

MATH 382 Lecture notes for 1-25-13

RADIUS OF CONVERGENCE

It's an easy but extremely important fact that every power series has associated with it a unique $0 \le R \le \infty$ such that

$$\begin{cases} |z| < R \implies \text{the power series converges absolutely at } z, \\ |z| > R \implies \text{the power series diverges at } z. \end{cases}$$

This is a quick result from what we have just proved.

There is actually a formula for R in general, but it won't be needed by us. Just to be complete, here's that formula:

$$\boxed{R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.}$$

<u>Useful observation</u>: suppose $|z| < R$, where R is the radius of convergence of $\sum_0^\infty a_n z^n$. Choose any z_1 such that $|z| < |z_1| < R$. Then from the preceding lecture we have an estimate

$$|a_n| \le C\,|z_1|^{-n}.$$

Now consider the quantity $n a_n z^n$:

NOTICE

$$|n a_n z^n| \le C n \left(\frac{|z|}{|z_1|}\right)^n.$$

Since $\frac{|z|}{|z_1|} < 1$, the real series

$$\sum_0^\infty n \left(\frac{|z|}{|z_1|}\right)^n$$

converges. We can actually appeal to the basic calculus <u>ratio test</u> to check this. Therefore,

$$\sum_0^\infty |n a_n z^n| < \infty.$$

Thus, not only does $\sum_0^\infty a_n z^n$ converge absolutely, but the series with <u>larger</u> coefficients na_n also converges absolutely ... remember, $|z| < R$.

CONCLUSION: $\boxed{\text{multiplying the coefficients } a_n \text{ of a power series by } n \text{ does not change the radius of convergence.}}$

<u>RATIO TEST</u> We just mentioned this result of basic calculus, namely, suppose that a series of positive numbers $\sum_0^\infty c_n$ has the property that

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \ell \text{ exists.}$$

Then

$$\begin{cases} \ell < 1 \Rightarrow \text{the series converges,} \\ \ell > 1 \Rightarrow \text{the series diverges.} \end{cases}$$

($\ell = 1$: no conclusion in general)

And now we apply this to power series $\sum_0^\infty a_n z^n$ with the property that

$$\boxed{\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \ell \text{ exists.}}$$

Then we can apply the ratio test to the series $\sum_0^\infty |a_n z^n|$, since

$$\lim_{n\to\infty} \frac{|a_{n+1} z^{n+1}|}{|a_n z^n|} = \ell |z|.$$

Thus

$$\begin{cases} \ell |z| < 1 \Rightarrow \text{convergence,} \\ \ell |z| > 1 \Rightarrow \text{divergence.} \end{cases}$$

That is, the radius of convergence of the power series equals

$$\boxed{R = \frac{1}{\ell}.}$$

EXAMPLES

- $\exp(z) = \sum_0^\infty \frac{z^n}{n!}$ $\qquad R = \infty$
- $\frac{1}{1-z} = \sum_0^\infty z^n$ $\qquad R = 1$
- $\sum_0^\infty n!\, z^n$ $\qquad R = 0$

Also, convergence for $|z| = R$ can happen variously:

$$\begin{cases} \sum_0^\infty z^n & \text{diverges for all } |z|=1, \\ \sum_1^\infty \frac{z^n}{n^2} & \text{converges for all } |z|=1, \\ \sum_1^\infty \frac{z^n}{n} & \text{diverges for } z=1, \text{ converges for all other } |z|=1. \end{cases}$$

we don't actually know this at the present time in this course

SIMPLE PROPERTIES OF POWER SERIES

Let $f(z) = \sum_0^\infty a_n z^n$ have radius of convergence R_1,
$g(z) = \sum_0^\infty b_n z^n$ have radius of convergence R_2.

$\boxed{\text{SUM}}$ $\quad f(z) + g(z) = \sum_0^\infty (a_n + b_n) z^n$ has radius of convergence $\geq \min(R_1, R_2)$.

$\boxed{\text{PRODUCT}}$ $\quad f(z)g(z) = \sum_0^\infty c_n z^n$ has radius of convergence $\geq \min(R_1, R_2)$,

where

$$c_n = \sum_{k=0}^n a_k b_{n-k}.$$

$\boxed{\text{DERIVATIVE}}$ For $|z|<R_1$, the function f has a complex derivative, and

$$f'(z) = \sum_1^\infty n a_n z^{n-1} \quad \ldots \text{ notice } \underline{\text{same}} \text{ radius of convergence.}$$

☆ We'll soon be able to prove the fact about products and this fact about $f'(z)$ with very little effort, almost no calculation involved. But I want to show you a direct proof for $f'(z)$. So let $|z|<R_1$ be fixed, and $h \in \mathbb{C}$ with small modulus, so that in particular

$$|z| + |h| < R_1.$$

Then we compute

$$f(z+h) - f(z) - h\sum_1^\infty n a_n z^{n-1} = \sum_1^\infty \left[a_n (z+h)^n - a_n z^n - n a_n z^{n-1} h \right]$$

$$= \sum_2^\infty a_n \left[(z+h)^n - z^n - n z^{n-1} h \right]$$

binomial formula

$$= \sum_{n=2}^\infty a_n \left[\sum_{k=0}^n \binom{n}{k} z^{n-k} h^k - z^n - n z^{n-1} h \right]$$

$$= \sum_{n=2}^\infty a_n \left[\sum_{k=2}^n \binom{n}{k} z^{n-k} h^k \right]$$

$$= h^2 \sum_{n=2}^\infty a_n \left[\sum_{k=2}^n \binom{n}{k} z^{n-k} h^{k-2} \right].$$

Divide by h:

$$\frac{f(z+h) - f(z)}{h} - \sum_1^\infty n a_n z^{n-1} = h \sum_{n=2}^\infty a_n \left[\sum_{k=2}^n \binom{n}{k} z^{n-k} h^{k-2} \right].$$

It follows easily that $f'(z)$ exists and equals $\sum_1^\infty n a_n z^{n-1}$.

Therefore, $\boxed{\text{every power series is holomorphic on its open disc of convergence.}}$

HW 5M SOLUTIONS

Let D be the open half plane

$$D = \{z \in \mathbb{C} \mid Re(z) > 0\}.$$

Let f be the function defined on D by $f(z) = z^2$. Of course, f is holomorphic.

1. Prove that f is a bijection of D onto a set $D' \subset \mathbb{C}$.

2. What is D'?

I'll use polar coordinates for D: $z = re^{i\theta}$, $0 < r < \infty$, $\underline{-\frac{\pi}{2} < \theta < \frac{\pi}{2}}$.

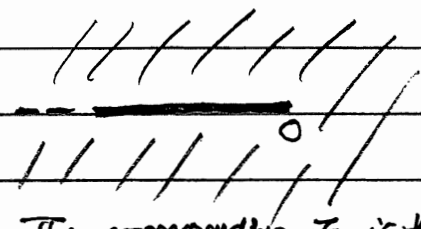
I've chosen $\arg z$

Then

$$z^2 = r^2 e^{2i\theta}, \quad -\pi < 2\theta < \pi.$$

Thus we read off that

$$D' = \underline{\mathbb{C} \sim \text{nonpositive real axis}}$$



and each $w \in D'$ can be written

$$w = \rho e^{i\varphi}$$

with $-\pi < \varphi < \pi$ and $\rho > 0$. The corresponding z is then

$$z = \sqrt{\rho}\, e^{i\varphi/2}.$$

BIJECTION

3. The inverse function f^{-1} maps D' onto D. We'll actually prove a general theorem asserting that <u>inverses of holomorphic functions are always holomorphic</u>. But in this problem I want you to prove directly that f^{-1} is holomorphic.

We've just seen that

$$f^{-1}(\rho e^{i\varphi}) = \sqrt{\rho}\, e^{i\varphi/2}.$$

Thus

$$\frac{\partial f^{-1}}{\partial \rho} = \frac{1}{2\sqrt{\rho}} e^{i\varphi/2} \quad \text{and} \quad \frac{\partial f^{-1}}{\partial \varphi} = \frac{i}{2}\sqrt{\rho}\, e^{i\varphi/2}$$

so we find

$$\frac{\partial f^{-1}}{\partial \rho} = \frac{1}{i\rho}\frac{\partial f^{-1}}{\partial \varphi}$$: the <u>Cauchy-Riemann eqn!</u>

(As a bonus,

$$(f^{-1})'(w) = e^{-i\varphi}\frac{\partial f^{-1}}{\partial \rho} = \frac{1}{2\sqrt{\rho}} e^{-i\varphi/2}$$

$$= \frac{1}{2}\frac{1}{f^{-1}(w)}.$$

i.e. $$\frac{d\sqrt{w}}{dw} = \frac{1}{2\sqrt{w}}.)$$

Jan 25(6)

4 For every real number $0 < a < \infty$ let L_a be the straight line

$$L_a = \{ z \in \mathbb{C} \mid Re(z) = a \}.$$

Prove that the images $f(L_a)$ are parabolas.

Now I'll use cartesian coordinates:

$$z = x+iy, \quad w = z^2 = u+iv.$$

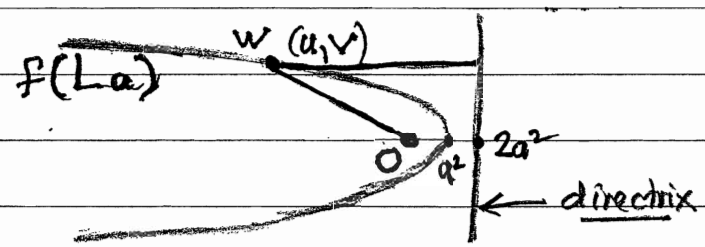
i.e.,

$$\begin{cases} u = x^2 - y^2, \\ v = 2xy. \end{cases}$$

L_a: $x = a$, so $u = a^2 - y^2$ and $v = 2ay$. Thus $y = \frac{v}{2a}$ and we have

$$\boxed{u = a^2 - \frac{v^2}{4a^2},}$$ ← the cartesian eqn. of a parabola. ← this parabola is $f(L_a)$

5 Prove that the *focus* of each parabola $f(L_a)$ is the origin.



$$|u+iv| = |w| = |z^2| = |z|^2 = x^2+y^2$$
$$= a^2+y^2 = 2a^2 - u,$$

so that

distance from w to O = distance from w to directrix.

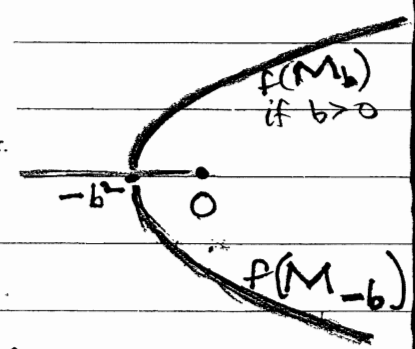
This shows that O is the focus.

6 For each real number b let M_b be the ray

$$M_b = \{ z \in \mathbb{C} \mid Im(z) = b \}.$$

Since f is conformal, the sets $f(M_b)$ and the parabolas $f(L_a)$ are orthogonal to one another.

Describe the sets $f(M_b)$.



M_b: $y = b$, so $u = x^2 - b^2$ and $v = 2bx$.

$b = 0$: $w = x^2$ gives the positive real axis.

$b \neq 0$: $u = \frac{v^2}{4b^2} - b^2$ and $\text{sign}(v) = \text{sign}(b)$:

half a parabola.

HW 6W due Jan 30, 2013

A power series centered at 0 is often called a <u>Maclaurin</u> series.

In the following excercises <u>simplify your answers</u> as much as possible.

1. Find the Maclaurin series for $\dfrac{1}{(1-z)^3}$.

2. Find the Maclaurin series for $\left(\dfrac{z}{3-z}\right)^2$.

3. Find the Maclaurin series for $e^z \sin z$.

4. Let $\omega = e^{2\pi i/3}$. $(1+\omega+\omega^2=0)$
   Find the Maclaurin series for $\dfrac{e^z+e^{\omega z}+e^{\omega^2 z}}{3}$.

5. Find explicitly $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(z-\pi i)^n}{n!}$.

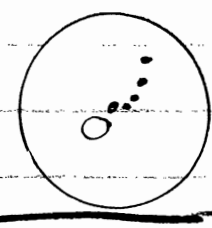
6. Find explicitly $\displaystyle\sum_{n=0}^{\infty} \frac{z^{5n}}{5^n}$.

MATH 382 Lecture notes for 1-28-13

More basic results about power series

First, a very simple theorem which will have profound consequences!

THEOREM: Suppose that $f(z) = \sum_0^\infty a_n (z-z_0)^n$ is a power series with a positive radius of convergence. And suppose that $f(z) = 0$ for an infinite sequence of points z converging to z_0.



Then $f = 0$. In other words, $a_n = 0$ for all n.

PROOF: We assume $z_0 = 0$ with no loss of generality.
Our proof is by contradiction, so we suppose that not all $a_n = 0$.
Then we have $a_N \neq 0$ for a smallest N, so that

$$f(z) = \sum_{n=N}^\infty a_n z^n$$
$$= z^N \sum_{n=N}^\infty a_n z^{n-N}$$
$$=: z^N g(z),$$

where $g(z)$ is the power series

$$g(z) = \sum_{k=0}^\infty a_{N+k} z^k$$
$$= a_N + a_{N+1} z + \cdots.$$

Then $f(z) = 0$ and $z \neq 0 \Rightarrow g(z) = 0$. Therefore our hypothesis implies that $g(z) = 0$ for an infinite sequence of points z converging to 0.
But $\lim_{z \to 0} g(z) = g(0) = a_N$. Thus $\underline{a_N = 0}$. Contradiction. QED

Taylor series

Again we suppose that $f(z) = \sum_0^\infty a_n (z-z_0)^n$ is a power series with positive radius of convergence. Then we observe

$$f(z_0) = a_0;$$
$$f'(z) = \sum_1^\infty n a_n (z-z_0)^{n-1}, \text{ so } f'(z_0) = a_1;$$
$$f''(z) = \sum_2^\infty n(n-1) a_n (z-z_0)^{n-2}, \text{ so } f''(z_0) = 2a_2.$$

In this manner we find

$$f^{(k)}(z_0) = k!\, a_k.$$

Therefore,

$$\boxed{f(z) = \sum_0^\infty \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n.}$$

The right side of this equation is called <u>the Taylor series of f centered at z_0</u>.

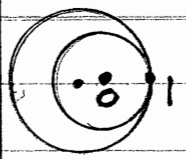
(If $z_0 = 0$, it's called the <u>Maclaurin</u> series of f.)

Changing center of power series

First, a couple of examples:

<u>Example 1</u> $f(z) = \sum_0^\infty z^n$ for $|z| < R = 1$, the geometric series.

Let's see about an expansion of $f(z)$ centered instead at $-\frac{1}{2}$.



Then we write

$$f(z) = \frac{1}{1-z} \quad \text{(sum of geometric series)}$$
$$= \frac{1}{\frac{3}{2} - (z+\frac{1}{2})}$$
$$= \frac{2}{3} \frac{1}{1 - \frac{z+\frac{1}{2}}{3/2}}$$
$$= \frac{2}{3} \sum_0^\infty \left(\frac{z+\frac{1}{2}}{3/2}\right)^n \quad \text{(a different geometric series)},$$

and this series converges in the disk $|z+\frac{1}{2}| < \frac{3}{2}$ ...

$$f(z) = \sum_0^\infty \frac{(z+\frac{1}{2})^n}{(\frac{3}{2})^{n+1}}.$$

<u>Example 2</u> $f(z) = \frac{1}{z}$, and we want to express this in a power series centered at $z_0 \neq 0$. Then as in the preceding example, we write

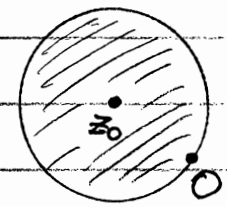
$$f(z) = \frac{1}{z_0 + (z - z_0)}$$

$$= \frac{1}{z_0} \frac{1}{1 + \frac{z - z_0}{z_0}}$$

$$= \frac{1}{z_0} \sum_0^\infty (-1)^n \left(\frac{z - z_0}{z_0}\right)^n \quad \text{(geometric series)}$$

$$= \sum_0^\infty \frac{(-1)^n}{z_0^{n+1}} (z - z_0)^n,$$

a Taylor series with radius of convergence $|z_0|$:



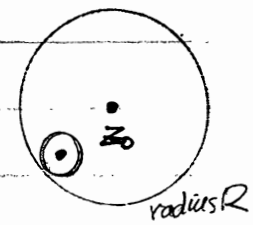
A very general <u>theorem</u>:

let $f(z) = \sum_0^\infty a_n (z - z_0)^n$ be a power series with radius of convergence R, and assume $|z_1 - z_0| < R$

Then $f(z) = \sum_0^\infty b_n (z - z_1)^n$

and the radius of convergence of this new series is

$$\geq R - |z_1 - z_0|.$$



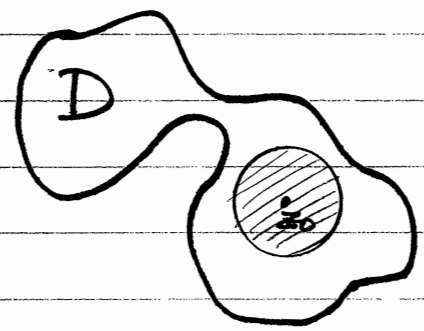
Although it is easy enough to prove this theorem with basic manipulations we already know, such a proof is tedious and boring. We'll soon be able to prove this theorem and many others with almost no effort at all!

These ideas lead us to an important <u>definition</u>:

Suppose f is a $\mathbb{C}$-valued function defined on an open subset $D \subset \mathbb{C}$, and suppose that for every $z_0 \in D$ we are able to write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all } |z - z_0| < R(z_0),$$

where $R(z_0)$ is some positive number. Then we say that f is (<u>complex</u>) <u>analytic</u> on D.



It's then quite clear that

every analytic function is holomorphic.

After we obtain <u>Cauchy's integral formula</u>, we'll see that the exact converse is valid:

<u>every holomorphic function is analytic</u>!

HW 6W SOLUTIONS

1. $\frac{1}{1-z} = \sum_0^\infty z^n$; differentiation $\Rightarrow \frac{1}{(1-z)^2} = \sum_1^\infty n z^{n-1}$.

once again, $\frac{2}{(1-z)^3} = \sum_2^\infty n(n-1) z^{n-2}$. Thus

$$\boxed{\frac{1}{(1-z)^3} = \sum_2^\infty \frac{n(n-1)}{2} z^{n-2}}$$

2. $\frac{1}{3-z} = \frac{1}{3}\frac{1}{1-\frac{z}{3}} = \sum_0^\infty \frac{z^n}{3^{n+1}}$ (geometric series).

Differentiation $\Rightarrow$ $\frac{1}{(3-z)^2} = \sum_1^\infty \frac{n z^{n-1}}{3^{n-1}}$.

Multiply by z^2: $\boxed{\left(\frac{z}{3-z}\right)^2 = \sum_1^\infty \frac{n z^{n+1}}{3^{n-1}}}$

3. $$e^z \sin z = e^z \frac{e^{iz} - e^{-iz}}{2i}$$

$$= \frac{1}{2i}\left(e^{(1+i)z} - e^{(1-i)z}\right)$$

$$= \frac{1}{2i}\sum_0^\infty \frac{(1+i)^n - (1-i)^n}{n!} z^n$$

$$= \frac{1}{2i}\sum_0^\infty \frac{\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^n - \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^n}{n!} z^n$$

$$= \frac{1}{2i}\sum_0^\infty 2^{\frac{n}{2}} \frac{e^{in\pi/4} - e^{-in\pi/4}}{n!} z^n$$

$$= \boxed{\sum_0^\infty 2^{\frac{n}{2}} \sin\frac{n\pi}{4} \frac{z^n}{n!}}$$

4. $e^{z}+e^{\omega z}+e^{\omega^2 z}$ has z^n coefficient equal to

$$\frac{1+\omega^n+\omega^{2n}}{n!}.$$

If $n \equiv 0 \pmod 3$, this is $\frac{1+1+1}{n!} = \frac{3}{n!}$ since $\omega^3=1$;

if $n \equiv 1 \pmod 3$, it is $\frac{1+\omega+\omega^2}{n!} = 0$;

if $n \equiv 2 \pmod 3$, it is $\frac{1+\omega^2+\omega^4}{n!} = 0$.

Thus
$$\frac{e^{z}+e^{\omega z}+e^{\omega^2 z}}{3} = \sum_{n\equiv 0 \pmod 3} \frac{1}{n!} z^n$$
$$= \boxed{\sum_{k=0}^{\infty} \frac{z^{3k}}{(3k)!}}$$

5.
$$\sum_{n=0}^{\infty} (-1)^n \frac{(z-\pi i)^n}{n!} = e^{-(z-\pi i)} = e^{\pi i} e^{-z}$$
$$= \boxed{-e^{-z}} \quad \underline{\text{for all } z \in \mathbb{C}} \qquad = -e^{-z}$$

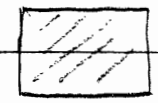
6. Geometric:
$$\sum_{0}^{\infty} \frac{z^{5n}}{5^n} = \sum_{0}^{\infty} \left(\frac{z^5}{5}\right)^n$$
$$= \boxed{\frac{1}{1-z^5/5}} \quad \underline{\text{for } |z|<5^{1/5}}$$

Although we didn't discuss this Monday, it fits in that subject matter. The principle involved here is based on simple single-variable calculus:

<u>Lemma</u> Suppose $f = f(z)$ has partial derivatives of first order which satisfy

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \quad \text{on a rectangle } (x_0, x_1) \times (y_0, y_1).$$

Then f is constant on that rectangle.



<u>Theorem</u> Suppose $D \subset \mathbb{C}$ is an open <u>connected</u> set and $D \xrightarrow{f} \mathbb{C}$ has partial derivatives of first order which satisfy

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \quad \text{on } D.$$

Then f is constant on D.

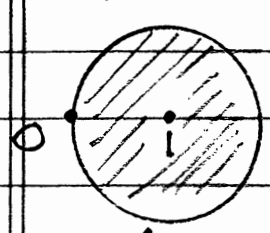
<u>Proof</u> By the lemma, f is constant on rectangles contained in D. Since D is <u>connected</u>, f is constant on D.

QED

<u>Corollary</u> Suppose $D \subset \mathbb{C}$ is an open connected set and $D \xrightarrow{f} \mathbb{C}$ is holomorphic on D with $f'(z) = 0$ for all $z \in D$. Then f is constant.

<u>Illustration</u>: For $|z| < 1$ the number $1 - z$ can be chosen to have $-\frac{\pi}{2} < \arg(1-z) < \frac{\pi}{2}$. Then

$$\frac{d}{dz} \log(1-z) = \frac{-1}{1-z} = -\sum_0^\infty z^n = -\frac{d}{dz} \sum_1^\infty \frac{z^n}{n}.$$



1-z in this disc

Thus $\log(1-z) + \sum_1^\infty \frac{z^n}{n}$ satisfies the hypothesis of the corollary for $|z| < 1$, and is thus constant. At $z = 0$ it equals 0. Therefore,

$$\log(1-z) = -\sum_1^\infty \frac{z^n}{n} \quad \text{for } |z| < 1.$$

Jan 28 (8)

EXAM 1 Due February 1, 2013

1. Find the unique Möbius transformation f which satisfies

$$f(1) = 1,$$
$$f(-1) = -1,$$
$$f(3) = 2.$$

2. Let f and g be two power series centered at 0, with radii of convergence R_1 and R_2, respectively.

a. Prove that if $R_1 < R_2$, then the power series $f + g$ has radius of convergence equal to R_1.

b. Give an example of f and g with $R_1 = R_2 = 1$, for which $f + g$ has radius of convergence 3.

3. Define the *Joukowski* function J by the equation

$$J(z) = \frac{z + \frac{1}{z}}{2} .$$

a. Prove that J is a bijection of the region $\{ z \in \mathbb{C} \mid |z| > 1 \}$ onto a region of the complex plane, and state exactly what that region is.

b. Let C_r be the circle with center 0 and radius $r > 1$. Show that J transforms this circle onto an ellipse whose foci are 1 and -1.

4. Find explicitly the sum of the power series

$$\sum_{n=1}^{\infty} n^2 z^n .$$

HONOR PLEDGE