COMPLEX ANALYSIS

CHAPTER III

INTEGRATION

Jan 30 — path integrals

HW 7M

Feb 1 — fundamental theorem of calculus

Cauchy integral theorem

Cauchy integral formula

HW 8W

Feb 4 — holomorphic $\Longleftrightarrow$ analytic

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Feb 13 — complex powers

HW 13M

MATH 382 Lecture notes for 1-30-13

INTEGRATION

Today we begin an absolutely fascinating journey of the study of holomorphic functions!

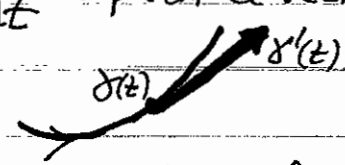
REVIEW OF VECTOR CALCULUS

The particular thing we need is called <u>path integration</u> or <u>line integration</u> or <u>contour integration</u>. It is based on curves in $\mathbb{R}^n$, which we'll typically denote by γ. These will need to be given a <u>parametrization</u> (at least in theory, if not explicitly) so that γ can be thought of as a function defined on an interval $[a,b] \subset \mathbb{R}$ with values in $\mathbb{R}^n$:

$$[a,b] \xrightarrow{\gamma} \mathbb{R}^n.$$

We'll need γ to be piecewise C^1. Its image in $\mathbb{R}^n$ may look something like this:

Notice that as t varies from a to b, $\gamma(t)$ moves in a definite direction. And $\gamma'(t) = \frac{d\gamma}{dt}$ represents a vector in $\mathbb{R}^n$ which is tangent to the curve.



For any $1 \le i \le n$ we then define the <u>path integral</u> of a function f along γ, in the x_i direction, as

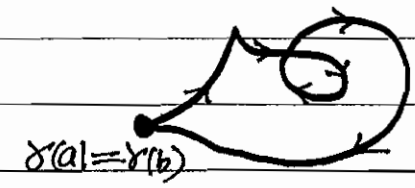
$$\int_\gamma f\,dx_i := \int_a^b f(\gamma(t))\,\gamma_i'(t)\,dt.$$

The chain rule shows that this result is independent of "reasonable"

changes of parametrization. But if we replace t by $-t$, the curve is traced in the opposite direction, so that

$$\int_{\substack{\text{REVERSED}\\ \gamma}} f\,dx_i = -\int_\gamma f\,dx_i .$$

A <u>loop</u> is a curve with $\gamma(a)=\gamma(b)$:

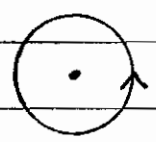


<u>Complex-valued f</u> No difficulty with this at all.

<u>Special notation for $\mathbb{R}^2$</u> Usually use x and y instead of x_1 and x_2.

<u>Example</u>

$$\int_{\substack{\text{CCW unit}\\ \text{circle}}} \frac{1}{z}\,dx = \int_0^{2\pi} \frac{1}{e^{i\theta}}\,d(\cos\theta)$$

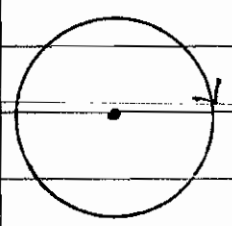


$$= \int_0^{2\pi} e^{-i\theta}(-\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} (\cos\theta - i\sin\theta)(-\sin\theta)\,d\theta$$

$$= 0 + i\int_0^{2\pi} \sin^2\theta\,d\theta = \pi i.$$

<u>Example</u> Let γ = clockwise circle with center O and radius r.



Then

$$\int_\gamma \frac{1}{z^2}\,dy = -\int_0^{2\pi} \frac{1}{(re^{i\theta})^2}\,d(r\sin\theta)$$

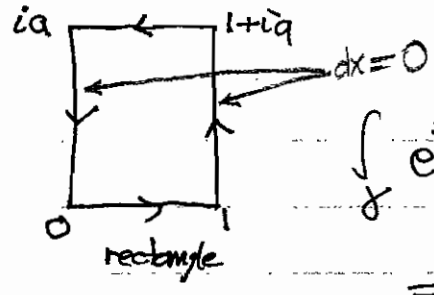
$$= -\frac{1}{r}\int_0^{2\pi} e^{-2i\theta}\cos\theta\,d\theta$$

$$= -\frac{1}{r}\int_0^{2\pi} e^{-2i\theta}\,\frac{e^{i\theta}+e^{-i\theta}}{2}\,d\theta$$

$$= -\frac{1}{2r}\int_0^{2\pi} \left(e^{-i\theta}+e^{-3i\theta}\right)d\theta$$

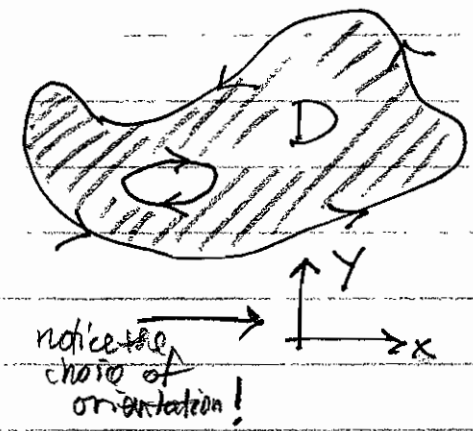
$$= 0.$$

<u>Example</u>



$$\int_\gamma e^z\,dx = \int_0^1 e^x\,dx + 0 + \int_1^0 e^{x+ia}\,dx + 0$$
$$= (1-e^{ia})\int_0^1 e^x\,dx$$
$$= (1-e^{ia})(e-1).$$

Of special importance for us is **Green's Theorem**:



If D is a "reasonably nice" bounded region, then we can consider ∂D, the boundary of D, as a <u>curve</u> or a union of curves, and we always give it the <u>orientation</u> or <u>direction</u> which keeps D on the left.

Then for a C^1 function f we have

$$\iint_D \frac{\partial f}{\partial x}\,dx\,dy = \int_{\partial D} f\,dy,$$
$$\iint_D \frac{\partial f}{\partial y}\,dx\,dy = -\int_{\partial D} f\,dx.$$

Usually these are presented as a single formula:

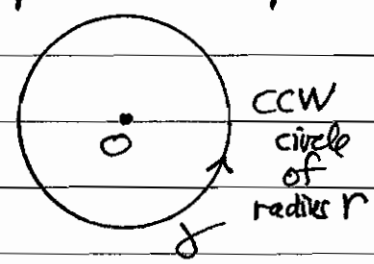
GREEN: $$\boxed{\iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\,dy = \int_{\partial D}(f\,dx + g\,dy).}$$

Remember: f and g are allowed to be complex-valued functions.

<u>Complex path integrals</u> Not only can the functions we are integrating be complex valued, but also we can integrate with respect to dz: just think $\underline{dz} = d(x+iy) = \underline{dx + i\,dy}$. Then we write

$$\int_\gamma f\,dz.$$

Most important example:



$$\int_\gamma \frac{1}{z}\,dz \quad \ldots \text{ parametrize with } z = re^{i\theta},\ 0 \le \theta \le 2\pi:$$

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{re^{i\theta}}\,d(re^{i\theta})$$

$$= \int_0^{2\pi} \frac{1}{re^{i\theta}}\left(rie^{i\theta}d\theta\right)$$

$$= \int_0^{2\pi} i\,d\theta = 2\pi i.$$

$$\boxed{\int_\gamma \frac{dz}{z} = 2\pi i}$$

<u>Special application of Green</u>: use a function f and $g = if$:

$$\iint_D \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)dx\,dy = \int_{\partial D} f\,dx + if\,dy.$$

Rewrite:

$$\boxed{\int_{\partial D} f\,dz = i\iint_D \left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right)dx\,dy.}$$

Jan 30 (5)

SOLUTIONS FOR EXAM 1

1. Find the unique Möbius transformation f which satisfies

$$f(1)=1,\quad f(-1)=-1,\quad f(3)=2.$$

A unique Möbius g sends $\begin{Bmatrix}1\\-1\\3\end{Bmatrix} \longrightarrow \begin{Bmatrix}0\\\infty\\1\end{Bmatrix}$ respectively:

$$g(z)=2\frac{z-1}{z+1}. \qquad g\sim\begin{pmatrix}2&-2\\1&1\end{pmatrix}$$

A unique Möbius h sends

$$\begin{Bmatrix}1\\-1\\2\end{Bmatrix}\to\begin{Bmatrix}0\\\infty\\1\end{Bmatrix}, \text{ resp.}$$

$$h(z)=3\frac{z-1}{z+1}. \qquad h\sim\begin{pmatrix}3&-3\\1&1\end{pmatrix}.$$

Then

$$f=h^{-1}\circ g\sim\begin{pmatrix}3&-3\\1&1\end{pmatrix}^{-1}\begin{pmatrix}2&-2\\1&1\end{pmatrix} \qquad \begin{pmatrix}1&3\\-1&3\end{pmatrix}\begin{pmatrix}2&-2\\1&1\end{pmatrix}$$

$$=\begin{pmatrix}5&1\\1&5\end{pmatrix}.$$

Thus

$$\boxed{f(z)=\frac{5z+1}{z+5}.}$$

2. Let f and g be two power series centered at 0, with radii of convergence R_1 and R_2, respectively.

a. Prove that if $R_1<R_2$, then the power series $f+g$ has radius of convergence equal to R_1.

b. Give an example of f and g with $R_1=R_2=1$, for which $f+g$ has radius of convergence 3.

a. Let R = radius of convergence of $f+g$. We know from class that $R\geq R_1$.

But also $f=(f+g)+(-g)$, so we also know from class that $R_1\geq\min(R,R_2)=R$ because $R_1<R_2$.

$\therefore \boxed{R=R_1}$

b. One example:

$$\underline{f=\frac{1}{1-z}} \quad\text{and}\quad \underline{g=\frac{1}{3-z}-\frac{1}{1-z}}.$$

Jan 30(6)

3. Define the *Joukowski* function J by the equation

$$J(z) = \frac{z + \frac{1}{z}}{2}.$$

Н. Е. Жуковский

a. Prove that J is a bijection of the region $\{ z \in \mathbb{C} \mid |z| > 1 \}$ onto a region of the complex plane, and state exactly what that region is.

b. Let C_r be the circle with center 0 and radius $r > 1$. Show that J transforms this circle onto an ellipse whose foci are 1 and -1.

a. <u>Injection</u>: if $J(z_1) = J(z_2)$, then

$$z_1 + \frac{1}{z_1} = z_2 + \frac{1}{z_2};$$

multiply by $z_1 z_2$: $\quad z_1^2 z_2 + z_2 = z_1 z_2^2 + z_1;$

$$z_1^2 z_2 - z_1 z_2 = z_1 - z_2;$$

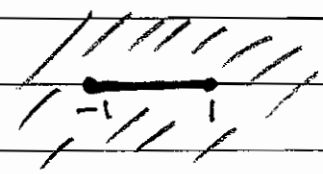
$$z_1 z_2 (z_1 - z_2) = z_1 - z_2;$$

$$(z_1 z_2 - 1)(z_1 - z_2) = 0.$$

But $|z_1| > 1$ and $|z_2| > 1 \Rightarrow |z_1 z_2| > 1 \Rightarrow z_1 z_2 - 1 \neq 0$.

Therefore $z_1 = z_2$. ✓

<u>Image</u> equals $\mathbb{C} \sim$ closed interval from -1 to 1:



Here's one proof. Let w belong to the described set, so that w is not a real number satisfying $-1 \leq w \leq 1$. Want to solve for z:

$$\frac{z + \frac{1}{z}}{2} = w, \quad |z| > 1.$$

That is, $\quad z^2 - 2wz + 1 = 0, \quad |z| > 1.$

This quadratic equation has two roots z_1 and z_2, and we have

$$z_1 z_2 = 1, \quad z_1 + z_2 = 2w.$$

If $|z_1| < 1$, then $|z_2| > 1$ and <u>z_2 is our solution.</u>

If both $|z_1| = 1$ and $|z_2| = 1$, then $z_1 = e^{i\theta}$ and $z_2 = e^{-i\theta}$ for a real number θ. But then

$$w = \frac{z_1 + z_2}{2} = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta,$$

a real number in the interval $[-1, 1]$. Contradiction!

b. $$|J(z)\pm 1| = \left|\frac{z+\frac{1}{z}}{2}\pm 1\right| = \frac{|z^2+1\pm 2z|}{2|z|}$$

$$= \frac{|z\pm 1|^2}{2r} = \frac{|z|^2 \pm 2\mathrm{Re}(z)+1}{2r} = \frac{r^2\pm 2\mathrm{Re}(z)+1}{2r}.$$

$\therefore$

$$|J(z)+1| + |J(z)-1| = \frac{r^2+2\mathrm{Re}(z)+1}{2r} + \frac{r^2-2\mathrm{Re}(z)+1}{2r}$$

$$= \frac{r^2+1}{r},$$

which is the same for all $z \in C_r$.

Thus $J(C_r)$ is an ellipse with foci at -1 and 1.
(The "focus definition" of ellipses.)

Alternate proof: write $z = re^{i\theta}$ and $J(z) = u+iv$. Then

$$u+iv = \frac{re^{i\theta}+\frac{1}{r}e^{-i\theta}}{2}$$

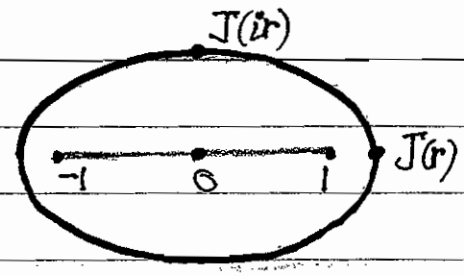
$$= \frac{r+\frac{1}{r}}{2}\cos\theta + i\frac{(r-\frac{1}{r})}{2}\sin\theta,$$

so that

$$\begin{cases} u = \frac{r+\frac{1}{r}}{2}\cos\theta, \\ v = \frac{r-\frac{1}{r}}{2}\sin\theta. \end{cases}$$

These are parametric equations for an ellipse:

$$\frac{u^2}{\left(\frac{r+\frac{1}{r}}{2}\right)^2} + \frac{v^2}{\left(\frac{r-\frac{1}{r}}{2}\right)^2} = 1.$$



As r varies from 1 to ∞, these ellipses cover $\mathbb{C} \sim [-1,1]$, giving another proof of the image $J(\{|z|>1\})$.

4. Find explicitly the sum of the power series
$$\sum_{n=1}^{\infty} n^2 z^n .$$

Start with geometric series
$$\frac{1}{1-z} = \sum_{0}^{\infty} z^n \qquad \text{for } |z|<1 .$$

Differentiate:
$$\frac{1}{(1-z)^2} = \sum_{1}^{\infty} n z^{n-1} .$$

Differentiate:
$$\frac{2}{(1-z)^3} = \sum_{1}^{\infty} n(n-1) z^{n-2} .$$

times z

times z^2

ADD

$$\frac{2z^2 + z(1-z)}{(1-z)^3} = \sum_{1}^{\infty} (n^2 - n + n) z^n ,$$

$$\therefore \quad \boxed{\frac{z+z^2}{(1-z)^3}} = \sum_{1}^{\infty} n^2 z^n .$$

MATHEMATICA →

In[1]:= $\sum_{n=1}^{\infty} n\hat{}2 * z\hat{}n$

Out[1]= $-\frac{z\,(1+z)}{(-1+z)^3}$

HW 7M due Feb 4, 2013

We know that there is a unique Möbius transformation f of $\widehat{\mathbb{C}}$ which satisfies

$$\begin{cases} f(0) = -1, \\ f(\infty) = 1, \\ f(i) = 0. \end{cases}$$

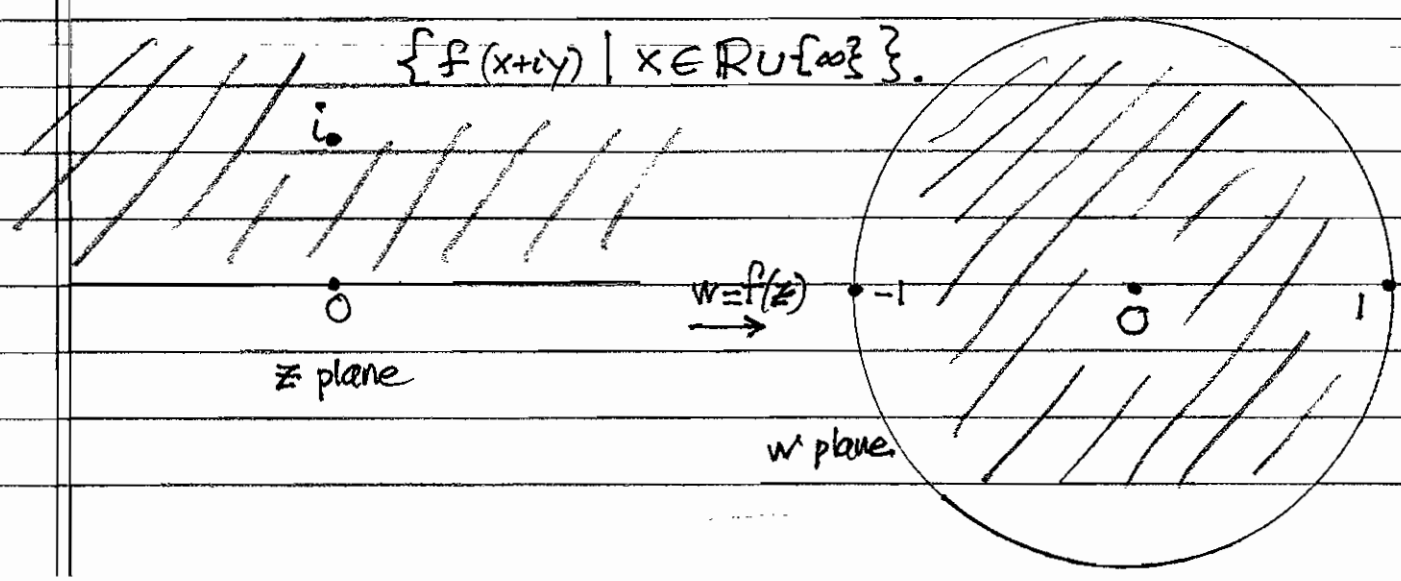
This Möbius function is called the <u>Cayley transform</u>.

1. Write explicitly $f(z) = \dfrac{az+b}{cz+d}$ (i.e. find a, b, c, d).

2. Prove that
$$f(\mathbb{R} \cup \{\infty\}) = \text{the unit circle.}$$

3. Prove that
$$f(\text{open upper half plane}) = \text{open unit disc.}$$

4. For several values of $y > 0$ sketch the image of the straight lines in the upper half plane
$$\{ f(x+iy) \mid x \in \mathbb{R} \cup \{\infty\} \}.$$



MATH 382 Lecture notes for 2-1-13

The fundamental theorem of calculus and path integrals

This simple fact is actually valid in $\mathbb{R}^n$ — vector calculus, but here is what it becomes in our complex situation:

Let γ be a path in $\mathbb{C}$ and f a holomorphic function. Then

(FTC) $$\boxed{\int_\gamma f'(z)\,dz = f(\text{final point of } \gamma) - f(\text{initial point of } \gamma).}$$

Proof: Let $\gamma = \gamma(t)$ for $a \le t \le b$. Then by definition

$$\int_\gamma f'(z)\,dz = \int_a^b f'(\gamma(t))\gamma'(t)\,dt$$

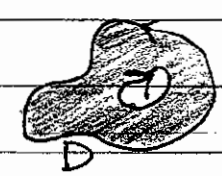
$$\overset{\text{CHAIN RULE}}{=} \int_a^b \frac{d}{dt}\Big(f(\gamma(t))\Big)\,dt$$

$$\overset{\text{FTC}}{=} f(\gamma(t))\Big|_a^b = f(\gamma(b)) - f(\gamma(a)).$$

QED

In the preceding lecture we used Green's theorem to prove that

$$\int_{\partial D} f\,dz = i\iint_D \left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right)dx\,dy.$$

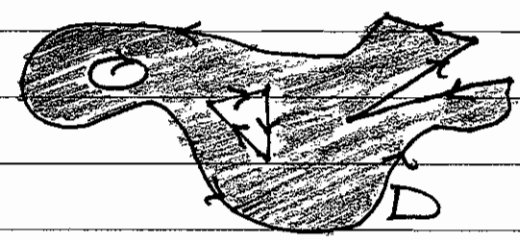


Notice that if f is holomorphic, then the Cauchy–Riemann equation, $\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$, gives a zero integrand on the right side of the Green equation, so that $\int_{\partial D} f\,dz = 0$. We now state this as a separate theorem:

THE CAUCHY INTEGRAL THEOREM

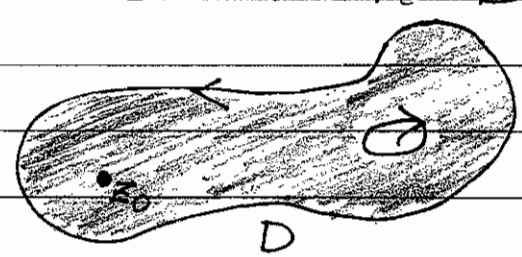
Suppose $D \subset \mathbb{C}$ is a "reasonably nice" bounded open set with boundary ∂D consisting of finitely many paths oriented with D on the left. Suppose f is a holomorphic function defined on an open set containing $D \cup \partial D$. Then

$$\int_{\partial D} f\,dz = 0.$$



We are now going to use this theorem to prove a truly amazing theorem, Cauchy's integral formula, which will be the basis for much of our subsequent study.

We assume the hypothesis exactly as above, but in addition we assume that a point $z_0 \in D$ is fixed... remember that D is open, so $z_0 \notin \partial D$:



We want to apply the Cauchy integral theorem to the function

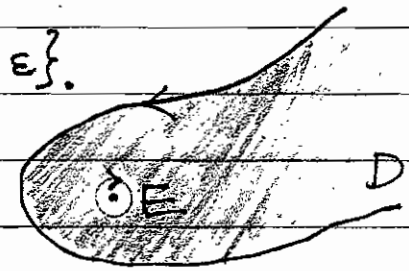
$$\frac{f(z)}{z - z_0},$$

but this function is not even defined at z_0.

The way around this difficulty is extremely clever, and also a strategy that is often used in similar situations not just in complex analysis, but also in partial differential equations and other places. It is the following

ruse: extract a small disc centered at z_0! Namely, let E be the closed disc of radius ε centered at z_0:

$$E = \{z \in \mathbb{C} \mid |z - z_0| \leq \varepsilon\}.$$



Then for sufficiently small ε $E \subset D$ since D is open, and we may apply the Cauchy theorem to the difference

$$D \sim E.$$

SAFETY DISC

We obtain

$$0 = \int_{\partial(D \sim E)} \frac{f(z)}{z - z_0}\, dz.$$

Now $\partial(D \sim E)$ is the disjoint union of ∂D and ∂E, so we have, using the correct orientation,

$$0 = \int_{\partial D} \frac{f(z)}{z - z_0}\, dz + \int_{\substack{\partial E \\ \text{CLOCKWISE} \\ \text{CIRCLE}}} \frac{f(z)}{z - z_0}\, dz.$$

Move the second integral to the left side and reverse the direction of the circle ∂E:

$$\int_{\substack{\partial E \\ \text{COUNTER-CLOCKWISE} \\ \text{CIRCLE}}} \frac{f(z)}{z - z_0}\, dz = \int_{\partial D} \frac{f(z)}{z - z_0}\, dz.$$

Fascinating equation! The right side is independant of ε, and thus so is the left side!

Parametrize ∂E: $z = z_0 + \varepsilon e^{i\theta}$, $0 \leq \theta \leq 2\pi$, so the left side equals

$$\int_0^{2\pi} \frac{f(z_0 + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\, \varepsilon i e^{i\theta}\, d\theta = i \int_0^{2\pi} f(z_0 + \varepsilon e^{i\theta})\, d\theta.$$

This can be rewritten as

$$2\pi i \text{ times } \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\varepsilon e^{i\theta})\,d\theta$$

$$= 2\pi i \text{ times } \underline{\text{the average of } f \text{ on } \partial E}.$$

This does not depend on ε! Yet it has a clear limit as $\varepsilon \to 0$, since f is continuous at z_0: namely, $2\pi i f(z_0)$. Therefore,

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z-z_0}\,dz.$$

BEWARE: notation change — z_0 replaced by z — z replaced by ZETA: ζ

Final result:

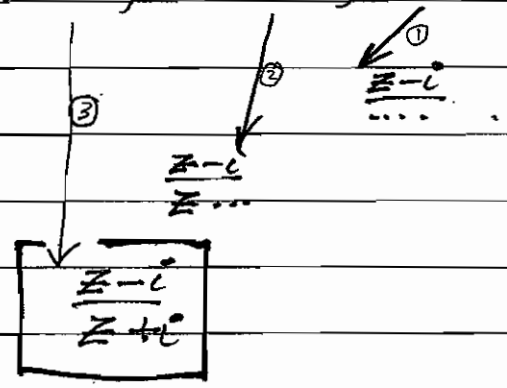
THE CAUCHY INTEGRAL FORMULA

Same hypothesis as the Cauchy integral theorem. Then for every $z \in D$

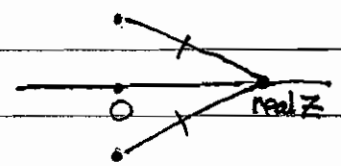
$$\boxed{f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,d\zeta.}$$

HW 7M SOLUTIONS

1. Cayley: Möbius with $f(0)=-1$, $f(\infty)=1$, $f(i)=0$.

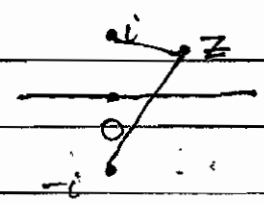


2.



For real z $|z-i|=|z+i|$, so $|f(z)|=1$. And $f(\infty)=1$. Möbius $\Rightarrow$ straight line gets mapped to line or circle, so we get unit circle

3.



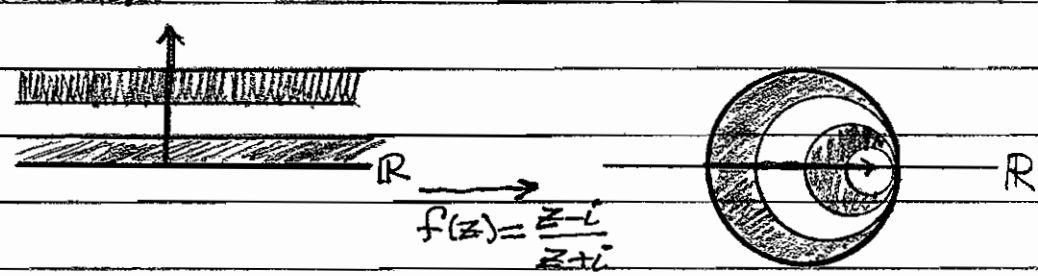
$\operatorname{Im} z>0 \Rightarrow |z-i|<|z+i|$

$\Rightarrow f(z) \in$ open unit disc.

Also, $\operatorname{Im}<0 \Rightarrow |f(z)|>1$.

As f is bijective, we get the entire open unit disc.

4. $f(\mathbb{R}+iy)$ must be a circle containing $f(\infty)=1$, contained in the unit disk, and therefore tangent to the unit circle. And $f(iy)=\frac{iy-i}{iy+i}=\frac{y-1}{y+1}$ is the other point of the circle on the real axis.



HW 8W due Feb 6, 2013

Give examples of two power series centered at 0 as follows:

$f(z)$ has radius of convergence 1,

$g(z)$ has radius of convergence 2,

$f(z)g(z)$ has radius of convergence 10.

All discipline for the moment for the moment seems not to be joyful, but sorrowful; yet to those who have been trained by it, afterwards it yields the peaceful fruit of righteousness.

Hebrews 12[11]

MATH 352 Lecture notes for 2-4-13

We derive very quickly many great consequences of the Cauchy integral formula.

1. Holomorphic functions are C^∞

This is rather stunning given that the definition of holomorphic required f to be of class C^1 and satisfy the Cauchy-Riemann equation. The key to this observation is that the dependence of $f(z)$ on z has been relegated to the simple function $\frac{1}{\zeta - z}$:

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

For $z \in D$ (open set) and $\zeta \in \partial D$, the function $\frac{1}{\zeta - z}$ is quite well behaved and we have for fixed ζ

$$\frac{d}{dz} \frac{1}{\zeta - z} = \frac{1}{(\zeta - z)^2}.$$

Therefore by performing $\frac{d}{dz}$ through the integral sign we obtain

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

We already knew $f'(z)$ existed, but now our same observation shows that $f'(z)$ has a complex derivative (we didn't know that before), and that

$$f''(z) = \frac{2}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^3} d\zeta.$$

Continuing in this manner, we see that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

QED

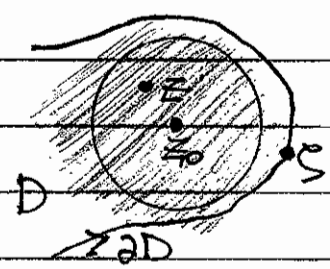
In particular,

2. f holomorphic $\Longrightarrow$ f' is holomorphic

Now we can also fulfill the promise made Jan 28(4):

3. Every holomorphic function is analytic

Once again, the key to this is the nature of $\frac{1}{\zeta - z}$. We establish a power series expansion in a disc centered at an arbitrary point $z_0 \in D$. As D is open, there exists $a > 0$ such that $|\zeta - z_0| \geq a$ for all $\zeta \in \partial D$. We then suppose that



$$|z - z_0| < a.$$

Looking for geometric series, we have

$$\frac{1}{\zeta - z} = \frac{1}{\underbrace{(\zeta - z_0)}_{\text{BIG}} - \underbrace{(z - z_0)}_{\text{SMALL}}}$$

$$= \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \quad \text{— modulus} < 1$$

$$= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

Since $\left|\frac{z-z_0}{\zeta-z_0}\right| \le \frac{|z-z_0|}{a} < 1$ for all $\zeta \in \partial D$, we have <u>uniform convergence</u> of the geometric series (rate of convergence same for all $\zeta \in \partial D$) and we conclude that

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} f(\zeta) \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}}\, d\zeta$$

$$= \sum_{n=0}^{\infty} c_n (z-z_0)^n,$$

where the coefficients are given by

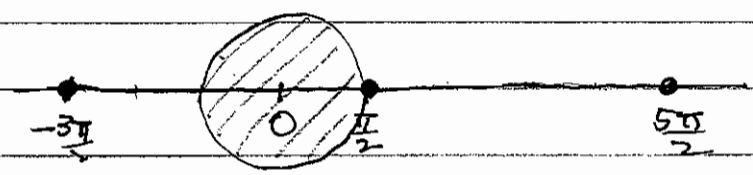
$$c_n = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta.$$

(By the way, notice from 1 that $c_n = \frac{1}{n!} f^{(n)}(z_0)$.)

Clearly, the radius of convergence of this power series is at least a... of course, it might be larger.

EXAMPLE $\frac{1}{1-\sin z}$ is holomorphic wherever $\sin z \neq 1$. And $\sin\frac{\pi}{2} = 1$, so HW 4W yields $\sin z = 1 \iff z = (2k+\frac{1}{2})\pi$.

We conclude that



$$\frac{1}{1-\sin z} = \sum_{0}^{\infty} c_n z^n \text{ with radius of convergence } \frac{\pi}{2}.$$

(Did not need to calculate any of the coefficients.)

Next, a converse to Cauchy's integral theorem:

4. Morera's theorem

Suppose f is a continuous function defined on an open set $D \subset \mathbb{C}$, with the property that for all loops γ contained in D,

$$\int_\gamma f(z)\,dz = 0.$$

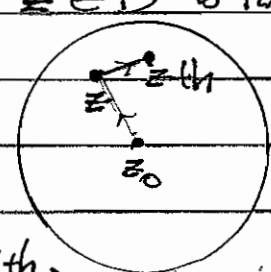
Then f is holomorphic.

(This theorem and its proof are similar to the result in vector calculus relating zero path integrals of a vector field to the vector field's having zero curl.)

Proof: This theorem is local in nature, so it suffices to prove it for the case in which D is a disk. Let $z_0 =$ center of D, and define the function on D

$$g(z) = \int_\gamma f(\zeta)\,d\zeta, \quad \text{where } \gamma = \text{any path in } D \text{ from } z_0 \text{ to } z.$$

Our hypothesis g arantees that $g(z)$ depends only on z, not on the choice of γ. Now assume $z \in D$ is fixed and $h \in \mathbb{C}$ is so small that $z+h \in D$:



then $g(z+h)$ can be calculated using the straight line from z_0 to z and then from z to $z+h$,

$$g(z+h) = g(z) + \int_z^{z+h} f(\zeta)\,d\zeta.$$

straight line segment

Parametrize the line segment from z to $z+h$ as $z+th$, $0 \le t \le 1$.

Then

$$g(z+h) - g(z) = \int_0^1 f(z+th)\,h\,dt$$

$$= h\int_0^1 f(z+th)\,dt.$$

Therefore

$$\frac{g(z+h)-g(z)}{h} = \int_0^1 f(z+th)\,dt.$$

Since f is continuous at z, the right side of this equation has limit $f(z)$ when $h \to 0$. Thus the left side has the same limit. We conclude that

$$g'(z) \text{ exists, and } g'(z) = f(z).$$

Since f is continuous, so is g'. Thus <u>g is holomorphic</u>. By 2, f is holomorphic.

QED

★ REMARK: the <u>proof</u> of Morera's theorem shows that the only hypothesis actually needed is that f be continuous and that in small discs contained in D,

$$\int_\gamma f\,dz = 0$$

for all <u>triangles</u> γ contained in the disk!

HW & W SOLUTION Here's one example:

$$f(z) = \frac{z-2}{z-1}, \quad g(z) = \frac{z-1}{(z-2)(z-10)}.$$

(We could calculate the Maclaurin series if we wished, using partial fractions:

$$f(z) = 1 + \frac{1}{1-z} = 1 + \sum_0^\infty z^n;$$

$$g(z) = \frac{-1/8}{z-2} + \frac{9/8}{z-10} = \frac{1/16}{1-z/2} - \frac{9/80}{1-\frac{z}{10}}$$

$$= \frac{1}{16}\sum_0^\infty \left(\frac{z}{2}\right)^n - \frac{9}{80}\sum_0^\infty \left(\frac{z}{10}\right)^n.)$$

HW 9F due Feb 8, 2013

$$\sec z \quad \left(:= \frac{1}{\cos z}\right)$$

This function is holomorphic in some disc centered at 0. Therefore it has a Maclaurin representation near 0.

1. Prove that only even terms z^{2n} are in this representation.

2. Find its radius of convergence.

★ 3. This expansion is customarily expressed in this form:

$$\sec z = \sum_{n=0}^{\infty} \frac{S_n z^{2n}}{(2n)!}.$$

Prove that all $S_n > 0$. The S_n's are called <u>secant</u> <u>numbers</u>.

Here are given $S_0, S_1, \dots, S_{16}$:

A000364 Euler (or secant or "Zig") numbers: e.g.f. (even powers only) sech(x)=1/cosh(x). 121
(Formerly M4019 N1667)

1, 1, 5, 61, 1385, 50521, 2702765, 199360981, 19391512145, 2404879675441, 370371188237525, 69348874393137901, 15514534163557086905, 4087072509293123892361, 1252259641403629865468285, 441543893249023104553682821, 177519391579539289436664789665 (list; graph; refs; listen; history; text; internal

MATH 382 Lecture notes for 2-6-13

$e^{z+w} = e^z e^w$ bi's

We gave Proof #1 the first day of the course. Now two more proofs.

Proof #2 For fixed $w \in \mathbb{C}$ consider the function

$$f(z) := e^{z+w} e^{-z}.$$

This holomorphic function has $f'(z) = e^{z+w}e^{-z} - e^{z+w}e^{-z} = 0$ by the product rule, so $f(z) =$ constant. This constant $= f(0) = e^w$. Thus

$$e^{z+w}e^{-z} = e^w \quad \text{for all } w \text{ and all } z.$$

When $w = 0$ we obtain $e^z e^{-z} = 1$, so that $e^{z+w} = e^w e^z$.

QED

Proof #3 • Let $w \in \mathbb{R}$ be fixed. Then the analytic function of z,

$$e^{z+w} - e^z e^w,$$

equals 0 for all real z from basic calculus. This occurrence of an infinity of zeros near $0 \Rightarrow$ the analytic function is 0: (see

$$e^{z+w} - e^z e^w = 0 \quad \text{for all } z \in \mathbb{C}, \text{ all } w \in \mathbb{R}.$$

• Now let $z \in \mathbb{C}$ be fixed. Then the analytic function of w,

$$e^{z+w} - e^z e^w$$

equals 0 for all real w, as we've just proved. Therefore as above it's 0 for all $w \in \mathbb{C}$.

QED

Basic estimates for complex integrals

a. Consider a complex-valued function $f = f(t)$ for $a \le t \le b$, and its integral

$$I := \int_a^b f(t)\,dt.$$

Write I in polar form,

$$I = |I| e^{i\theta} \quad \text{for some } \theta \in \mathbb{R}.$$

Then

$$
\begin{aligned}
|I| &= e^{-i\theta} I \\
&= e^{-i\theta}\int_a^b f(t)\,dt && \text{(def. of } I\text{)} \\
&= \int_a^b e^{-i\theta} f(t)\,dt && (e^{-i\theta} \text{ is a constant}) \\
&= \operatorname{Re}\int_a^b e^{-i\theta} f(t)\,dt && \text{(it's already real)} \\
&= \int_a^b \operatorname{Re}\left(e^{-i\theta} f(t)\right) dt && \text{(def. of complex integration)} \\
&\le \int_a^b |e^{-i\theta} f(t)|\,dt \\
&= \int_a^b |f(t)|\,dt.
\end{aligned}
$$

Thus we have

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$

b. Path integrals: let the curve γ be parametrized as $\gamma = \gamma(t)$ for $a \le t \le b$. Then assume $|f(z)| \le C$ for all $z = \gamma(t)$. Then

$$
\begin{aligned}
\left|\int_\gamma f(z)\,dz\right| &= \left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right| \\
&\le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt && \text{(by a)} \\
&\le C\int_a^b |\gamma'(t)|\,dt \\
&= CL, \text{ where } L = \text{length of } \gamma.
\end{aligned}
$$

Thus

$$\left|\int_\gamma f\,dz\right| \le \max_\gamma |f| \cdot \text{length of } \gamma.$$

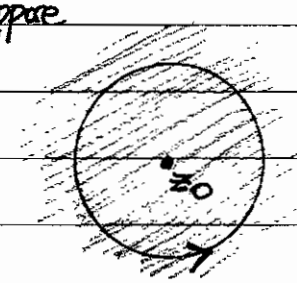
P.S. More generally, we see that $\left|\int_\gamma f\,dz\right| \le \int_\gamma |f|\,|dz|$, where

$$|dz| = |dx + i\,dy| = \sqrt{(dx)^2 + (dy)^2} = d(\text{arc length}).$$

Now we continue with consequences of the Cauchy integral formula. Last time we listed 4 of them, so we now come to

5. Mean value property of holomorphic functions

Let f be holomorphic on an open set $D \subset \mathbb{C}$ and suppose a closed disc $|z - z_0| \leq r$ is contained in D. Then the Cauchy formula gives in particular



$$f(z_0) = \frac{1}{2\pi i} \int_{\substack{|\zeta - z_0| = r \\ (\text{ccw})}} \frac{f(\zeta)}{\zeta - z_0} d\zeta .$$

The usual parametrization $\zeta = z_0 + re^{i\theta}$ yields

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \text{ , the average of } f \text{ on the circle.}$$

Before the next result, here's an important bit of terminology:

> an **entire** function (or **entire holomorphic** function) is a function which is defined and holomorphic on all of $\mathbb{C}$.

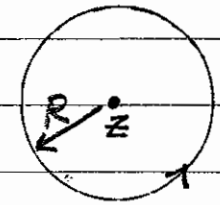
6. Liouville's theorem

An entire function which is bounded must be constant.

Proof: Let $f = f(z)$ be entire and suppose $|f(z)| \leq C$ for all $z \in \mathbb{C}$, where C is a constant.

Let $z \in \mathbb{C}$ be arbitrary, and apply Cauchy's formula using the disk with center z and radius R. Then from the preceding lecture we have

$$f'(z) = \frac{1}{2\pi i}\int_{\substack{|\zeta - z| = R \\ \text{CCW}}} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta.$$



Therefore we estimate

$$|f'(z)| \leq \frac{1}{2\pi}\int_{|\zeta - z| = R} \frac{C}{|\zeta - z|^2}\, |d\zeta|$$

$$= \frac{1}{2\pi}\int_{|\zeta - z| = R} \frac{C}{R^2}\, |d\zeta|$$

$$= \frac{1}{2\pi}\frac{C}{R^2} \cdot \text{length of circle}$$

$$= \frac{1}{2\pi}\frac{C}{R^2} \cdot 2\pi R$$

$$= \frac{C}{R}.$$

Simply let $R \to \infty$ to conclude that $f'(z) = 0$. Thus $f' = 0$ on all of $\mathbb{C}$, so f is constant.

QED

During class we also talked about <u>harmonic</u> functions. These in general are functions u defined on $\mathbb{R}^n$ which satisfy Laplace's equation

$$\nabla^2 u = 0.$$

In a standard orthonormal coordinate system, this equation is

$$\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0.$$

Holomorphic functions are harmonic. For the Cauchy–Riemann equation

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} \Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{1}{i}\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{i}\frac{\partial^2 f}{\partial y \partial x} = \frac{1}{i}\frac{\partial}{\partial y}\left(\frac{1}{i}\frac{\partial f}{\partial y}\right)$$

$$= -\frac{\partial^2 f}{\partial y^2},$$

so that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

HW 9F SOLUTIONS

$\sec z \quad \left(:= \frac{1}{\cos z}\right)$

This function is holomorphic in some disc centered at 0. Therefore it has a Maclaurin representation near 0.

1. Prove that only even terms z^{2n} are in this representation.

2. Find its radius of convergence.

★ 3. This expansion is customarily expressed in this form:

$$\sec z = \sum_{n=0}^{\infty} \frac{S_n z^{2n}}{(2n)!}.$$

Prove that all $S_n > 0$.

1. $\sec(-z) = \sec(z)$, so the Maclaurin series can have only even terms.

2. $\cos z = 0 \iff z = (n+\frac{1}{2})\pi$. Therefore $\sec z$ is holomorphic for $|z| < \frac{\pi}{2}$. It cannot be holomorphic in a larger open disk since it is not holomorphic in a neighborhood of $\frac{\pi}{2}$. $\therefore$ radius of convergence $= \pi/2$

3. $(\sec z)' = \sec z \tan z$, and $(\tan z)' = \sec^2 z$. Then we calculate the derivatives of S like this:

$$S' = ST$$

$$S'' = ST(T) + S(S^2) = S^3 + ST^2$$

$$S''' = 3S^2(ST) + 2ST(S^2) + ST(T^2)$$
$$= 5S^3T + ST^3$$

etc.

In general,

$$S^{(n)} = \text{linear combination of } T^{n-2k}S^{2k+1}$$
$$\text{for } 0 \le k \le \tfrac{n}{2},$$

with coefficients which are <u>positive</u> integers.

(Easy proof by induction.)

Evaluate at $z=0$:

$$S^{(n)}(0) = 0 \text{ if } n \text{ is odd}$$
$$> 0 \text{ if } n \text{ is even.}$$

Since $\sec z$ has Maclaurin series $\sum_0^\infty \frac{S^{(n)}(0)}{n!} z^n$

$$= \sum_0^\infty \frac{S^{(2n)}(0)}{(2n)!} z^{2n},$$

we see that

$$S_n = S^{(2n)}(0) > 0.$$

QED

Can we prove these numbers are alternately 1 and 5 (mod 10)?

A000364 Euler (or secant or "Zig") numbers: e.g.f. (even powers only) sech(x)=1/cosh(x). (Formerly M4019 N1667) 121

1, 1, 5, 61, 1385, 50521, 2702765, 199360981, 19391512145, 2404879675441, 370371188237525, 69348874393137901, 15514534163557086905, 4087072509293123892361, 1252259641403629865468285, 441543893249023104553682821, 177519391579539289436664789665 (list; graph; refs; listen; history; text; internal

MATH 382 Lecture notes for 2-8-13

Actually, this is a **revised** version of what we did Friday, for I completely forgot to give a careful proof of the famous so-called

Fundamental theorem of algebra

Let P be a polynomial with complex coefficients and positive degree. Then there exists $z \in \mathbb{C}$ such that $P(z) = 0$.

Proof: We suppose to the contrary that for all $z \in \mathbb{C}$, $P(z) \neq 0$. Normalize P to be "monic" — that is,

$$P(z) = z^N + c_1 z^{N-1} + \cdots + c_N,$$

where $N \geq 1$. Then

$$\lim_{z \to \infty} \frac{P(z)}{z^N} = 1.$$

The function $\frac{1}{P}$ is entire because the denominator is never 0, and

$$\lim_{z \to \infty} \frac{z^N}{P(z)} = 1,$$

so we see that

$$\lim_{z \to \infty} \frac{1}{P(z)} = 0.$$

Therefore $\frac{1}{P}$ is a bounded entire function. Aha! **Liouville's theorem** implies that it is constant! Therefore $P(z)$ is constant. That's a contradiction. **QED**

REMARK: Since $P(z_1) = 0$ for some z_1, it's simple polynomial algebra which shows that the polynomial $P(z)$ is divisible by the polynomial $z - z_1$: $P(z) = (z - z_1)Q(z)$, where Q is a

polynomial of one less degree than P. If Q has positive degree, then again we conclude that for some z_2, $Q(z) = (z - z_2)R(z)$, where R is again a polynomial. Continuing in this way we have a factorization of P into linear factors:

$$P(z) = c \prod_{k=1}^{N} (z - z_k).$$

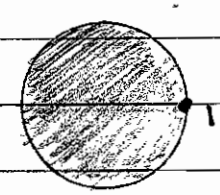
(Some z_k's may be repeated, of course.)

Later this semester we'll give a much different proof of the FTA in which the complete factorization will appear instantaneously!

Before we continue with consequences of the Cauchy integral formula, we pause to rethink the holomorphic function $\frac{1}{1-z}$. For $|z|<1$ we can simply write

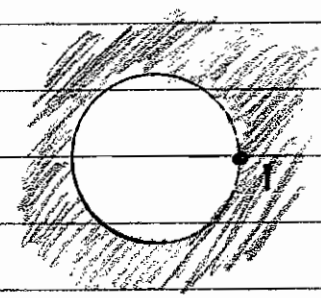
$$\frac{1}{1-z} = \sum_{0}^{\infty} z^n, \text{ the geometric series.}$$

This equation is valid $\Leftrightarrow |z|<1$.



Now suppose $|z|>1$. Then $1-z$ is dominated by z, so we write

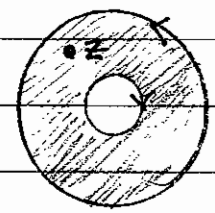
$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{-z}\,\frac{1}{1-\frac{1}{z}} \\ &= -\frac{1}{z}\sum_{0}^{\infty}\left(\frac{1}{z}\right)^n \\ &= -\sum_{0}^{\infty}\frac{1}{z^{n+1}} \\ &= -\sum_{1}^{\infty}\frac{1}{z^n},\end{aligned}$$



valid $\Leftrightarrow |z|>1$.

The procedure we have just reduced is useful in the following more general situation:

suppose f is holomorphic in an open set D which contains a closed annulus $r_1 \le |z| \le r_2$. For $r_1 < |z| < r_2$ we then employ the Cauchy integral formula to write $f(z)$ in terms of path integrals along $|z| = r_2$ counterclockwise and along $|z| = r_1$ clockwise:



$$f(z) = \frac{1}{2\pi i} \int_{\substack{|\zeta| = r_2 \\ CCW}} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\substack{|\zeta| = r_1 \\ CW}} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

- For $|\zeta| = r_2$ we write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \frac{1}{1 - \frac{z}{\zeta}} = \sum_{n=0}^{\infty} \frac{z^n}{\zeta^{n+1}}$$

so that the corresponding integral becomes

$$\frac{1}{2\pi i} \sum_{n=0}^{\infty} z^n \int_{\substack{|\zeta| = r_2 \\ CCW}} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta.$$

- For $|\zeta| = r_1$ we write

$$\frac{1}{\zeta - z} = \frac{-1}{z} \frac{1}{1 - \frac{\zeta}{z}} = -\sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}}$$

so that the corresponding integral becomes

$$-\frac{1}{2\pi i} \sum_{n=0}^{\infty} z^{-n-1} \int_{\substack{|\zeta| = r_1 \\ CW}} f(\zeta) \zeta^n d\zeta.$$

We can of course change the sign by performing the path integral the opposite direction.

We also change the dummy index n in the latter series by $-n-1=k$, so that k ranges from $-\infty$ to -1, with the result being

$$\frac{1}{2\pi i}\sum_{k=-\infty}^{-1} z^k \int_{\substack{|\zeta|=r_1\\ CCW}} \frac{f(\zeta)}{\zeta^{k+1}}\,d\zeta.$$

One more adjustment: the function $\frac{f(\zeta)}{\zeta^{n+1}}$ is holomorphic in the complete annulus $r_1 \le |\zeta| \le r_2$, so its path integral over a circle of radius r is <u>independent</u> of r, thanks to Cauchy's integral theorem. We therefore obtain our final result,

$$\star \qquad f(z)=\sum_{-\infty}^{\infty} c_n z^n, \text{ for } r_1<|z|<r_2,$$

where

$$c_n=\frac{1}{2\pi i}\int_{\substack{|\zeta|=r\\ CCW}} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta \quad (r_1\le r\le r_2).$$

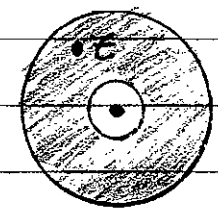
<u>TERMINOLOGY</u>: a series of the form $\star$, containing z^n for both positive and negative indices n, is called a **Laurent series.**

We now formulate what we have accomplished. As usual, we may immediately generalize to an arbitrary center z_0 instead of 0:

7. Laurent expansion theorem

Let $0 \le R_1 < R_2 \le \infty$, and assume that f is a holomorphic function in the open annulus centered at z_0:

$$R_1 < |z - z_0| < R_2 .$$



Then for all z in this annulus

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n ,$$

where c_n is given by

$$c_n = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta ,$$

and r is any radius satisfying $R_1 < r < R_2$.

Here's an important quick corollary:

8. Riemann's removable singularity theorem

Let f be a holomorphic function defined in a "punctured" disc $0 < |z - z_0| < R$, and assume f is bounded. Then there is a limit $f(z_0) := \lim_{z \to z_0} f(z)$ and the resulting function is holomorphic in the full disc $|z - z_0| < R$.

Proof: Suppose $|f(z)| \le C$ for $0 < |z - z_0| < R$. Apply the Laurent expansion theorem with $R_1 = 0$ and $R_2 = R$. Then for any index $n \le -1$, we can estimate c_n this way: for any $0 < r < R$,

$$|c_n| = \left| \frac{1}{2\pi i} \int_{|z-z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right|$$

$$\leq \frac{1}{2\pi} \frac{C}{r^{n+1}} \cdot \text{length of circle}$$

$$= \frac{C}{r^n}.$$

But when $r \to 0$, $C/r^n \to 0$ since $n < 0$. Thus $c_n = 0$ for all $n < 0$. Therefore we have the result that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad \text{for} \quad 0 < |z - z_0| < R.$$

$\leftarrow$ notice!

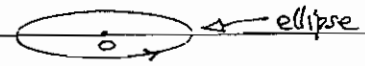
Clearly then, $\lim_{z \to z_0} f(z) = c_0$ and if we define $f(z_0) = c_0$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad \text{for} \quad |z - z_0| < R.$$

QED

Feb 8(7)

HW 10M SOLUTIONS

1. 

$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z} = 1$, where γ is the ellipse.

Manipulate this equation to prove that for all $a>0, b>0$

$$\boxed{\int_0^{2\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{2\pi}{ab}.}$$

Parametrize the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $x=a\cos t, y=b\sin t$.

That is,

$$z = a\cos t + ib\sin t.$$

Then

$$1 = \frac{1}{2\pi i}\int_0^{2\pi} \frac{-a\sin t + ib\cos t}{a\cos t + ib\sin t}\,dt$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{(-a\sin t + ib\cos t)(a\cos t - ib\sin t)}{|a\cos t + ib\sin t|^2}\,dt$$

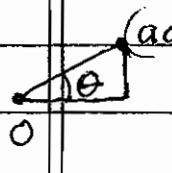
$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{(b^2-a^2)\sin t\cos t + i(ab\cos^2 t + ab\sin^2 t)}{a^2\cos^2 t + b^2\sin^2 t}\,dt$$

$$= \frac{1}{2\pi i}\left[0 + i\int_0^{2\pi}\frac{ab\,dt}{a^2\cos^2 t + b^2\sin^2 t}\right]$$

$$= \frac{ab}{2\pi}\int_0^{2\pi}\frac{dt}{a^2\cos^2 t + b^2\sin^2 t}.$$

QED

(P.S. This is actually an elementary result. For we can use the usual polar angle θ,



$\tan\theta = \frac{b}{a}\tan t$. Then $\sec^2\theta\, d\theta = \frac{b}{a}\sec^2 t\, dt$, so

$$\frac{dt}{a^2\cos^2 t + b^2\sin^2 t} = \frac{\sec^2 t\, dt}{a^2 + b^2\tan^2 t} = \frac{\frac{a}{b}\sec^2\theta\, d\theta}{a^2 + a^2\tan^2\theta} = \frac{1}{ab}\,d\theta.)$$

2. Liouville's theorem extended

Suppose that f is an entire function with the property that for all z

$$|f(z)| \leq c_1 + c_2 |z|^N,$$

where c_1 and c_2 are constants and N is a nonnegative integer. Prove that f is a polynomial of degree $\leq N$.

HINT: Use a Liouville type argument for the $N+1$ order derivative $f^{(N+1)}$.

We have

$$f(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where

$$b_n = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta,$$

valid for any $R > 0$. Then we estimate

$$|b_n| \leq \frac{1}{2\pi} \frac{c_1 + c_2 R^N}{R^{n+1}} \cdot 2\pi R$$

$$= \frac{c_1 R + c_2 R^{N+1}}{R^{n+1}}.$$

If $n \geq N+1$, then $n+1 \geq N+2$ so that

$$|b_n| \leq \lim_{R \to \infty} \frac{c_1 R + c_2 R^{N+1}}{R^{n+1}} = 0.$$

Then

$$f(z) = \sum_{n=0}^{N} b_n z^n.$$

QED

Feb 8(9)

HW 11W due Feb 13, 2013

The Bernoulli numbers

1. Show that the function of z given as $\dfrac{z}{e^z-1}$

has a removable singularity at the origin.

2. Therefore this function has a Maclaurin expansion, which we write in this form:

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n .$$

The B_n's are called the **Bernoulli** numbers

Find the radius of convergence of this series.

3. Use the equation $z = (e^z-1)\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$ to derive a recursion formula for the B_n's:

$$B_0 = 1,$$
$$B_1 = -\tfrac{1}{2},$$
$$\sum_{n=0}^{k-1} \binom{k}{n} B_n = 0 \quad \text{for } k \geq 2.$$

4. Prove that $B_n = 0$ for all odd $n \geq 3$.

HINT: examine $\dfrac{z}{e^z-1} + \dfrac{z}{2}$.

MATH 382 Lecture notes for 2-11-13

Isolated singularities

Let $z_0 \in \mathbb{C}$ be fixed, and suppose f is a function which is holomorphic for $0 < |z - z_0| < R$. Then f is said to have a singularity at z_0, simply because $f(z_0)$ is undefined. We actually say that f has an isolated singularity at z_0, since f is actually holomorphic in the disc $|z - z_0| < R$ except at z_0 (where it is undefined).

We then know that f has a Laurent expansion of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \text{ valid for } 0 < |z - z_0| < R.$$

There is a convenient classification of isolated singular points according to the appearance of c_n with $n < 0$ in the Laurent expansion. They are divided into 3 distinct categories as follows:

R = removable singularities, meaning that for all $n < 0$, $c_n = 0$.

P = poles, meaning that some $c_n \neq 0$ with $n < 0$, but there are only finitely many such c_n ... all the remaining c_n with $n < 0$ are 0.

E = essential singularities, meaning that $c_n \neq 0$ for infinitely many $n < 0$.

It is of utmost importance to have a complete understanding of these categories, so we devote the rest of this lecture to this.

REMOVABLE SINGULARITIES

In this case the Laurent expansion is

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \quad \text{for } 0<|z-z_0|<R.$$

The right side of this equation defines a function analytic for the entire disc $|z-z_0|<R$. Therefore, we <u>remove</u> the singularity of f <u>defining</u> $f(z_0) = c_0$.

Of course, we have the great **removable singularity theorem** of Riemann, which asserts that if we assume only that f is <u>bounded</u> near z_0, then its singularity at z_0 is removable. Thus we have these equivalent situations:

- the singularity at z_0 is removable
- f is bounded near z_0
- $\lim_{z \to z_0} f(z)$ exists and is in $\mathbb{C}$

Except in the trivial case that $f=0$, not all $c_n \neq 0$. Say that $c_N \neq 0$ with $N \geq 0$ minimal. Then we may write

$$\begin{aligned} f(z) &= c_N (z-z_0)^N + \text{higher order terms} \\ &= (z-z_0)^N g(z), \end{aligned}$$

where g is holomorphic and $g(z_0) \neq 0$. We then say that f <u>has a zero at z_0 of order N</u>. (Of course, $N=0$ is allowed.)

POLES In this case there exists $N < 0$ such that $c_N \neq 0$ but all c_n before that are 0. Therefore we may write

$$f(z) = \sum_{n=N}^{\infty} c_n (z-z_0)^N$$

$$= (z-z_0)^N \sum_{k=0}^{\infty} c_{N+k} (z-z_0)^k$$

$$= (z-z_0)^N g(z),$$

where g is holomorphic for $|z-z_0|<R$ and $g(z_0) \neq 0$. When then say that f <u>has a pole at z_0 of order</u> $-N$... in this case $-N \geq 1$.

We then have these equivalent situations:

- the singularity at z_0 is a pole
- $\lim_{z \to z_0} f(z) = \infty$.

(The former of those implies the latter, but we'll soon prove the reverse implication.)

EXAMPLES

- $\csc z$ has a pole at 0 of order 1
- $\sec z$ has a pole at $\frac{\pi}{2}$ of order 1
- $\frac{1}{e^z-1}$ has a pole at 0 of order 1
- $\frac{z}{e^z-1}$ has a removable singularity at 0
- $\frac{1}{z(e^{z^2}-1)}$ has a pole at 0 of order 3

ESSENTIAL SINGULARITIES

The Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

is neither of the first two kinds: z_0 is neither a removable singularity nor a pole.

In this situation the behavior of f as $z \to z_0$ is quite interesting:

Casorati–Weierstrass theorem

Suppose f has an essential singularity at z_0. Then for any $w \in \hat{\mathbb{C}}$ there exists a sequence $z_1, z_2, \cdots$ such that

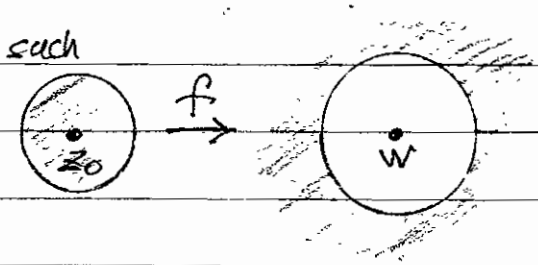
$$\lim_{k\to\infty} z_k = z_0 \quad \text{and} \quad \lim_{k\to\infty} f(z_k) = w.$$

Proof: We proceed by contradiction. That is, we suppose there exists $w \in \hat{\mathbb{C}}$ for which no sequence $\{z_k\}$ exists with

$$z_k \to z_0 \quad \text{and} \quad f(z_k) \to w.$$

This means that for z near z_0, $f(z)$ cannot be arbitrarily close to w... in other words, $f(z)$ must stay a positive distance away from w.

- If $w = \infty$, this means that $f(z)$ must be bounded for z near z_0. The removable singularity situation holds, so z_0 is a removable singularity for f. Contradiction.
- If $w \in \mathbb{C}$, then there exists $r > 0$ such that for z sufficiently near z_0, $|f(z) - w| > r$.



Then consider the function $\frac{1}{f(z)-w}$ in this neighborhood of z_0. It is bounded (by $\frac{1}{r}$) and thus its singularity at z_0 is removable. That is, it agrees with a holomorphic function near z_0. If this function has a zero of order $N \geq 0$ at z_0, it may be written as

$$(z-z_0)^N g(z),$$

where g is holomorphic and nonzero. Therefore

$$f(z)-w = (z-z_0)^{-N} \frac{1}{g(z)}.$$

As $\frac{1}{g(z)}$ is holomorphic, this equation shows that the Laurent series for f has no terms $(z-z_0)^n$ for $n \leq -N$. Contradiction.

QED

DISCUSSION This classification into the 3 types of isolated singularities is quite definitive and complete. However, as wonderful as the Casorati–Weierstrass theorem is, it doesn't come close to the much more profound result known as **Picard's great theorem:** if f has an isolated essential singularity at z_0, then for every $w \in \mathbb{C}$ with at most one exception, there exists a sequence $z_k \to z_0$ such that $f(z_k) = w$ for all $k = 1, 2, 3, \cdots$.

(The example $e^{1/z}$ has the exception $w = 0$.)

This theorem is "beyond the scope of our course."

HW 11W SOLUTIONS

1. Show that the function of z given as $\frac{z}{e^z-1}$ has a removable singularity at the origin.

Really easy, for $e^z-1=\sum_1^\infty \frac{z^n}{n!} = z\sum_1^\infty \frac{z^{n-1}}{n!}$, so

$$\frac{e^z-1}{z} = \sum_1^\infty \frac{z^{n-1}}{n!}$$ gives the removal of the singularity of $\frac{e^z-1}{z}$ at the origin. As the right side equals 1 at $z=0$, its reciprocal is holomorphic near 0. Thus $\frac{z}{e^z-1}$ is equal to this reciprocal (except at 0), which is holomorphic near 0.

2. Therefore this function has a Maclaurin expansion, which we write in this form:

$$\frac{z}{e^z-1} = \sum_{n=0}^\infty \frac{B_n}{n!} z^n.$$

The B_n's are called the **Bernoulli** numbers

Find the radius of convergence of this series.

$e^z-1=0 \iff z=2n\pi i$, so $e^z-1 \neq 0$ for $|z|<2\pi$, except at 0. Thus $\frac{z}{e^z-1}$ is holomorphic for $|z|<2\pi$. As its singularity at $2\pi i$ is not removable, the radius of convergence is exactly $\mathbf{2\pi}$.

3. Use the equation $z=(e^z-1)\sum_{n=0}^\infty \frac{B_n}{n!}z^n$ to derive a recursion formula for the B_n's:

$$B_0=1,$$
$$B_1=-\tfrac{1}{2},$$
$$\sum_{n=0}^{k-1}\binom{k}{n}B_n=0 \quad \text{for } k\geq 2.$$

We have $$z = \sum_{k\geq 1}\frac{z^k}{k!}\cdot\sum_{n\geq 0}\frac{B_n}{n!}z^n.$$

The power series on the left side equals the power series on the right side near 0. Thus they have the same coefficients. We simply equate them:

coefficient of z^m on the right side is equal to

$$\sum_{\substack{k\geq 1\\ n\geq 0\\ k+n=m}} \frac{B_n}{k!\,n!}.$$

$$m=1 \Rightarrow \quad 1 = \frac{B_0}{1!\,0!} = B_0.$$

$$m\geq 2 \Rightarrow \quad 0 = \sum_{\substack{n\geq 0\\ m-n\geq 1}} \frac{B_n}{(m-n)!\,n!} = \sum_{n=0}^{m-1} \frac{B_n}{(m-n)!\,n!}.$$

Multiply by $m!$ to get

$$0 = \sum_{n=0}^{m-1} \binom{m}{n} B_n.$$

In particular, $m=2 \Rightarrow 0 = \binom{2}{0}B_0 + \binom{2}{1}B_1 = 1+2B_1$, so $B_1 = -\frac{1}{2}$.

> **4.** Prove that $B_n = 0$ for all odd $n\geq 3$.
>
> HINT: examine $\frac{z}{e^z-1} + \frac{z}{2}$.

So indeed calculate

$$\frac{z}{e^z-1} + \frac{z}{2} = \frac{z}{2}\,\frac{2+(e^z-1)}{e^z-1} = \frac{z}{2}\,\frac{e^z+1}{e^z-1}$$

$$= \frac{z}{2}\,\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}},$$

which is

$$\frac{z}{2}\coth\left(\frac{z}{2}\right) \ldots \text{ an even function.}$$

That is,

$$\frac{z}{e^z-1} - B_1 z = \sum_{\substack{n=0\\ n\neq 1}}^{\infty} \frac{B_n}{n!} z^n \quad \text{is even.}$$

$\therefore$ its odd coefficients $B_3, B_5, B_7, \ldots$ are all 0.

QED

HW 12 F due Feb 15, 2013

1. Show that $\sinh z = w$ has a solution z for every w. Do this by deriving a "formula" for z in terms of w. This formula will involve a choice of <u>square root</u> and a choice of <u>log</u> but don't worry about these details at the present time.

2. Do the same for the equation $\tanh z = w$, but notice that there's one exception (actually, two) for w.

3. The function $\sin \frac{1}{z}$ has an essential singularity at 0. Verify directly for this function the truth of the great Picard theorem.

MATH 382 Lecture notes for 2-13-13

Complex powers

The goal is to devise a reasonable definition of z^α when α is allowed to be complex. Though it makes no sense to "raise z to the power α," we still use that terminology.

WARNING: when dealing with this subject it's very important not to use the notation e^z in the usual way, but instead to use the terminology from the first day of class,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

A good way to figure out what our definition should is the use of desired properties of logarithm, namely

$$\log(z^\alpha) = \alpha \log z.$$

Then we use the "inverse" of log to come to our definition:

$$\boxed{z^\alpha := \exp(\alpha \log z).}$$

This makes some sense as long as $z \neq 0$, so we'll always make that assumption. Of course, z^α is usually not a unique complex number, due to the ambiguity in $\log z$. For other values could be

$$\exp(\alpha(\log z + 2n\pi i)) = \exp(\alpha \log z)\exp(2n\pi i \alpha).$$

This will be independent of the integer $n \Leftrightarrow \alpha$ is an integer, and then z^α has its usual meaning.

Now we list the properties of this definition.

1. If α is an integer, z^α has it's usual meaning.
For all other $\alpha \in \mathbb{C}$, z^α is ambiguous, no matter what z is.

In particular, $\boxed{z^0 = 1.}$

2. $1^\alpha = \exp(\alpha \log 1) = \exp(\alpha(0 + 2n\pi i))$, so

$$\boxed{1^\alpha \text{ has all the values } \exp(2n\pi i\alpha).}$$

Notice especially that $e^\alpha = \exp(\alpha)\exp(2n\pi i\alpha)$ is ambiguous.

3. $i^i = \exp(i \log i) = \exp(i(0 + \frac{i\pi}{2} + 2n\pi i)) = \exp(-\frac{\pi}{2} - 2n\pi)$,

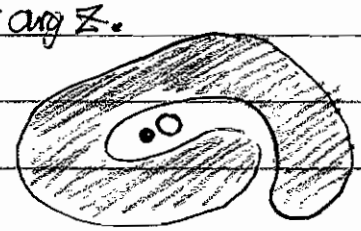
so

i^i has all the values $\exp\left(\left(\text{even integer} - \frac{1}{2}\right)\pi\right)$
(all are <u>real</u> numbers).

4. $\boxed{z^\alpha z^\beta = z^{\alpha+\beta}}$ provided the same $\log z$ is used in all 3 places it appears

5. $\boxed{z^\alpha w^\alpha = (zw)^\alpha \text{ — sort of true: be careful!}}$

6. In open sets $\subset \mathbb{C}$ which do not contain 0 and which do not "wind around 0," $\log z$ can be defined in terms of a continuous value for $\arg z$.



Then $\log z$ becomes a holomorphic function, as we know, so also the composite function z^α is holomorphic.

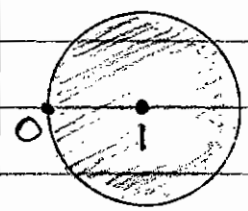
And we compute its derivative by the chain rule:

$$(z^\alpha)' = \exp(\alpha \log z)\,(\alpha \log z)'$$
$$= z^\alpha \frac{\alpha}{z},$$

and we write

$$\boxed{\frac{dz^\alpha}{dz} = \alpha z^{\alpha-1}}$$ (same $\log z$ on each side).

7. Taylor series



As in the above discussion, we take $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$ in this disc. Then we have inductively for $n = 0, 1, 2, \cdots$

$$\left(\frac{d}{dz}\right)^n z^\alpha = \alpha(\alpha-1)\cdots(\alpha-n+1)\, z^{\alpha-n}.$$

In particular, at $z=1$ we find $\alpha(\alpha-1)\cdots(\alpha-n+1)$ (since $1^\alpha = 1$).
So we obtain the Taylor series

$$z^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}(z-1)^n \quad \text{for } |z-1|<1.$$

That coefficient is given this notation

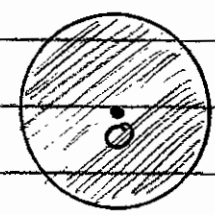
$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!},$$

and is still called a <u>binomial coefficient</u>:

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{1} = \alpha, \quad \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}, \text{ etc.}$$

Replacing z by $1+z$ yields a "binomial formula",

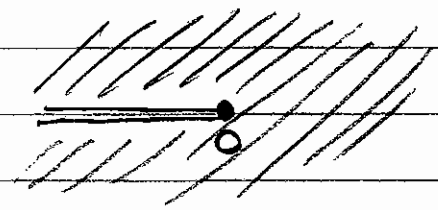
$$\boxed{(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n} \quad \text{for } |z|<1.$$



(If $\alpha = 0, 1, 2, \cdots$ this series is <u>finite</u>, going only from $0 \le n \le \alpha$. It's a polynomial, and this result is the classical <u>binomial formula</u>. Otherwise, the radius of

convergence equals 1.)

PRINCIPAL DETERMINATION OF ARG & LOG: this is what we say when we are in the open set $\mathbb{C} \sim (-\infty, 0]$:



$-\pi < \arg z < \pi$.

$1^x = 1$, $\log x$ is real for $0 < x < \infty$

HW 12F SOLUTIONS

1. <u>Solving $\sinh z = w$ for z</u>. We want $\frac{e^z - e^{-z}}{2} = w$, or

$$e^{2z} - 1 = 2we^z;$$
$$e^{2z} - 2we^z - 1 = 0.$$

Quadratic formula:

$$e^z = w \pm \sqrt{w^2+1},$$

two possible square roots. But we need to see that $w + \sqrt{w^2+1} \neq 0$; i.e., $w \neq -\sqrt{w^2+1}$; i.e., $w^2 \neq w^2+1$. Correct!

Thus

$$\underline{z = \text{any value of } \log\left(w + \sqrt{w^2+1}\right)}.$$

For short, I would just say

$$\boxed{\operatorname{arc\,sinh} w = \log\left(w + \sqrt{w^2+1}\right).}$$

2. <u>$\tanh z = w$</u> This time we want $\frac{\sinh z}{\cosh z} = w$. i.e.,

$$\frac{e^z - e^{-z}}{e^z + e^{-z}} = w;$$

$$\frac{e^{2z} - 1}{e^{2z} + 1} = w;$$

$$e^{2z} = \frac{w+1}{-w+1}.$$

Restriction: $\underline{w \neq \pm 1}$. Then

$$2z = \log\left(\frac{w+1}{-w+1}\right).$$

Loosely speaking,

$$\boxed{\operatorname{arctanh} w = \frac{1}{2}\log\frac{w+1}{w-1}.}$$

3. We want $\sin\frac{1}{z} = w$, where w is arbitrary and $z \to 0$. So for any w choose one z_0 such that $\sin\frac{1}{z_0} = w$ (using #1). Then

$$\sin\left(\frac{1}{z_0} + 2k\pi\right) = w \text{ for any } k \in \mathbb{Z}.$$

So take

$$z_k = \frac{1}{\frac{1}{z_0} + 2k\pi}.$$

Then

$$\underline{\lim_{k\to\infty} z_k = 0} \quad \text{and} \quad \underline{\sin\frac{1}{z_k} = w}.$$

QED

HW 13M due Feb 18, 2013

Prove this polynomial identity:

$$\binom{\alpha+\beta}{n} = \sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n-k}.$$

HINT: <u>very</u> easy problem!

BRANCH POINTS

All these holomorphic functions which somehow involve $\log z$ have a definite type of "singular" behavior near 0. But these are clearly not isolated singularities. For these functions are not actual functions (single-valued) in any region which includes all z satisfying $0<|z|<\varepsilon$.

Instead, we say that these functions have a branch point at 0. That's a well-chosen descriptive word, for as we follow their behavior on a loop surrounding 0 they can exhibit a change because of the change in a continuous determination of $\arg z$.

For instance, $z^{1/2}$ has 2 values, $z^{1/3}$ has 3 values, $z^{m/n}$ has n values, assuming the integers m and n have no common prime factor. However, if α is irrational, z^{α} has infinitely many values.

We say that the above functions have branching of order 2, 3, n, and ∞, respectively.

You can imagine that things could become more and more complicated. For instance, think about $z^{\sqrt{z}}$ near the origin.

Of course, we can also see branch points at other points, such as in the function $(z-1)^{1/2}+(z+1)^{1/2}$, which has branch points at 1 and -1. Or $(z^2+1)^{1/2}$ with branch points at i and $-i$.