COMPLEX ANALYSIS

CHAPTER IV

RESIDUES (PART 1)

Feb 15 residues

HW 14F

Feb 18 evaluation of definite integrals

Feb 20 evaluating, continued

Feb 22 evaluations, continued

HW 15W

Mar 4 principal value integrals

HW 16 F

Mar 6 integrals involving exponentials (and trig functions)

EXAM 2

Mar 8 trig functions, continued

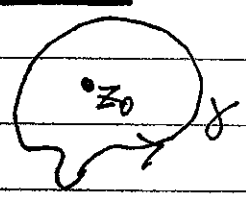
HW 17W

MATH 382 Lecture notes for F 2-15-13

RESIDUES

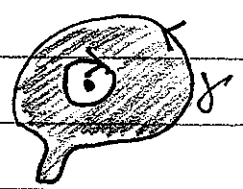
We are preparing to do some truly amazing things with our theory, but first we need an important definition. This is all in the context of a holomorphic function with an isolated singularity at z_0. Let f be such a function. We are then going to define a complex number based on this situation, but we do it in 3 separate ways, and we'll discover that these 3 ways yield the same number.

Definition 1: Let γ be any small loop surrounding z_0 in the CCW sense: then our number equals



$$\boxed{\frac{1}{2\pi i}\int_\gamma f(z)\,dz.}$$

This number does not depend on γ, thanks to Cauchy's integral theorem. For surround z_0 with a very small circle: in the region between γ and this circle we have from Cauchy's theorem



$$\int_\gamma f(z)\,dz + \int_{\substack{\text{CW}\\ \text{circle}}} f(z)\,dz.$$

Thus

$$\int_\gamma f(z)\,dz = \int_{\substack{\text{CCW}\\ \text{circle}}} f(z)\,dz,$$

so the left side does not depend on γ.

Definition 2: Consider the Laurent expansion of f near z_0:

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-z_0)^n.$$

Then our number equals

$$\boxed{c_{-1}.}$$

That is, we focus our attention on the Laurent expansion =

$$f(z) = \cdots + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \cdots$$

and it's c_{-1} we use.

This agrees with the first definition since

$$\int_\gamma f(z)\,dz = \sum_{-\infty}^{\infty} c_n \int_\gamma (z-z_0)^n\,dz = c_{-1}\int_\gamma \frac{dz}{z-z_0}$$
$$= 2\pi i\, c_{-1}.$$

<u>Definition **3**</u>: This definition relies on trying to integrate $f(z)$ as an "indefinite integral" near z_0. The trouble is precisely with the term $\frac{c_{-1}}{z-z_0}$. For

$$\int f(z)\,dz = \sum_{n \neq -1} c_n \frac{(z-z_0)^{n+1}}{n+1} + \int \frac{c_{-1}}{z-z_0}\,dz.$$

undefined!

Thus the number we want is the unique $a \in \mathbb{C}$ such that

$$\boxed{f(z) - \frac{a}{z-z_0} = \text{the derivative of a holomorphic function in the region } 0 < |z-z_0| < r, \text{ some } r > 0.}$$

Definition: The <u>residue of f at z_0</u> is the number defined in all 3 of the above definitions. We denote it as

$$\boxed{\mathrm{Res}(f, z_0).}$$

We now list a number of properties and examples.

1. If f is holomorphic at z_0 (i.e., z_0 is a <u>removable</u> singularity of f, then
$$\mathrm{Res}(f, z_0) = 0.$$

2. $$\mathrm{Res}\left(\frac{1}{z-z_0}, z_0\right) = 1. \quad \text{(Most basic case.)}$$

3. $$\mathrm{Res}\left((z-z_0)^n, z_0\right) = 0 \quad \text{if } n \in \mathbb{Z},\ n \neq -1.$$

4. $$\mathrm{Res}\left(e^{\frac{1}{z}}, 0\right) = 1.$$

5. $$\mathrm{Res}\left(\sin\frac{1}{z}, 0\right) = 1.$$

6. $$\mathrm{Res}\left(\cos\frac{1}{z}, 0\right) = 0.$$

7. If f is an <u>even</u> function,
$$\mathrm{Res}(f, 0) = 0.$$

8. Suppose f has a <u>simple pole</u> (i.e., pole of order 1) at z_0. Then we have
$$f(z) = \frac{c_{-1}}{z-z_0} + \sum_{n=0}^{\infty} c_n (z-z_0)^n,$$

so that

$$(z-z_0)f(z) = c_{-1} + (z-z_0)\sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

so that the left side has limit c_{-1} as $z \to z_0$:

SIMPLE POLE $$\boxed{\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z-z_0)f(z).}$$

9. A corollary is now the following, which is <u>the</u> <u>handiest</u> <u>special</u> <u>case</u>!
Suppose we know that

$$f(z) = \frac{a(z)}{b(z)},$$

where both numerator and denominator are holomorphic at z_0, and $b(z_0) = 0$, $b'(z_0) \neq 0$. Then z_0 is a simple pole for f, and our previous result yields

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{(z-z_0)a(z)}{b(z)}$$

$$= \lim_{z \to z_0} \frac{a(z)}{\frac{b(z) - \cancel{b(z_0)}^{\,0}}{z - z_0}}$$

$$= \frac{a(z_0)}{b'(z_0)}.$$

For the record,

$$\boxed{\operatorname{Res}\left(\frac{a(z)}{b(z)}, z_0\right) = \frac{a(z_0)}{b'(z_0)}} \quad \text{if } b(z_0) = 0,\ b'(z_0) \neq 0.$$

10. $\operatorname{Res}(\cot z, 0) = 1$ since $\cot z = \frac{\cos z}{\sin z}$ and at 0 $\cos 0 = 1$, $\sin 0 = 0$, $\sin'(0) = 1$.
More generally,

$$\boxed{\operatorname{Res}(\cot z, n\pi) = 1.} \qquad \left(\frac{\cos n\pi}{\cos n\pi}\right)$$

11. $$\boxed{\operatorname{Res}(\csc z, n\pi) = (-1)^n.}$$

12. $\operatorname{Res}\left(\frac{1}{e^z-1}, 0\right) = 1.$

13. $\operatorname{Res}\left(\frac{z}{e^z-1}, 0\right) = 0.$

14. Let's compute the residue at 0 of $\frac{1}{a-z}e^{1/z}$, where $a \neq 0$.

None of our easy examples apply, so we resort to series:

$$\frac{1}{a-z} = \frac{1}{a}\frac{1}{1-\frac{z}{a}} = \sum_0^\infty \frac{z^n}{a^{n+1}},$$

$$e^{\frac{1}{z}} = \sum_0^\infty \frac{1}{n!\,z^n}.$$

Multiply those series and look for the $\frac{1}{z}$ terms:

$$\frac{1}{a}\frac{1}{1!} + \frac{1}{a^2}\frac{1}{2!} + \frac{1}{a^3}\frac{1}{3!} + \cdots.$$

This equals $-1 + \sum_{k=0}^\infty \frac{1}{a^k k!}$, so

$$\operatorname{Res}\left(\frac{1}{a-z}e^{1/z}, 0\right) = e^{\frac{1}{a}} - 1.$$

Incidentally,

$$\operatorname{Res}\left(\frac{1}{a-z}e^{1/z}, a\right) = -e^{\frac{1}{a}}.$$

Now we come to a major theorem. Before stating it, let us be sure we completely understand the context.

As in the Cauchy integral theorem, we deal with a "nice" bounded open set $D \subset \mathbb{C}$, whose boundary ∂D consists of finitely many curves. We also assume that f is holomorphic on an open set containing $D \cup \partial D$ except for finitely many isolated singularities $z_1, \dots, z_n$, all contained in the open set D.

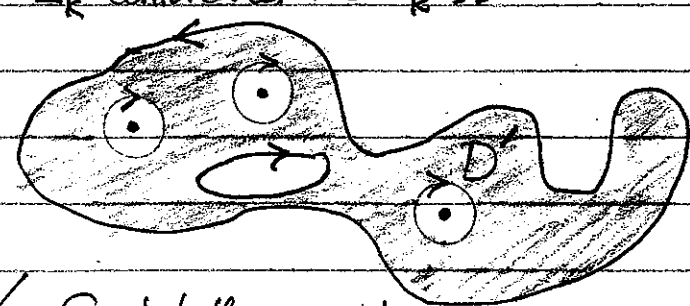
Then we have the

RESIDUE THEOREM

$$\frac{1}{2\pi i}\int_{\partial D} f(z)\,dz = \sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$

The proof is an easy application of the Cauchy integral theorem if we first remove from D small discs E_k centered at the z_k's. Let the resulting open set be denoted

$$D' = D \sim \bigcup_{k=1}^{n} E_k.$$



Since f is holomorphic on D', Cauchy's theorem yields

$$0 = \frac{1}{2\pi i}\int_{\partial D'} f(z)\,dz.$$

But $\partial D' = \partial D \cup \bigcup_{k=1}^{n} \partial E_k$, so we obtain

$$0 = \frac{1}{2\pi i}\int_{\partial D} f(z)\,dz + \sum_{k=1}^{n} \frac{1}{2\pi i}\int_{\partial E_k} f(z)\,dz.$$

clockwise

But

$$\frac{1}{2\pi i}\int_{\partial E_k \text{ clockwise}} f(z)\,dz = -\frac{1}{2\pi i}\int_{\partial E_k \text{ counterclockwise}} f(z)\,dz$$

$$= -\operatorname{Res}(f, z_k). \quad \text{(def. of residue)}$$

This proves the theorem.

QED

HW 13M SOLUTION

$$(1+z)^{\alpha+\beta} = (1+z)^{\alpha}(1+z)^{\beta}$$

is a valid identity if $|z|<1$ and we use the principal value of $\arg(1+z)$ in all 3 places.

In terms of Maclaurin series,

$$\sum_{n=0}^{\infty}\binom{\alpha+\beta}{n}z^n = \sum_{j=0}^{\infty}\binom{\alpha}{j}z^j \cdot \sum_{k=0}^{\infty}\binom{\beta}{k}z^k.$$

So equate the coefficients of z^n on the two sides of this equation:

$$\binom{\alpha+\beta}{n} = \sum_{j+k=n}\binom{\alpha}{j}\binom{\beta}{k}.$$

i.e.,

$$\binom{\alpha+\beta}{n} = \sum_{j=0}^{n}\binom{\alpha}{j}\binom{\beta}{n-j}.$$

QED

HW 14 F due Feb 22, 2013

1. For any positive integer n, find all the singularities of the function $\dfrac{1}{z^n+1}$ and calculate all the corresponding residues. Also, compute the sum of all the residues.

2. Suppose that f has a pole at z_0 of order $\leq N$. The function $(z-z_0)^N f(z)$ has a removable singularity at z_0. Prove that

$$\operatorname{Res}(f, z_0) = \left.\frac{\left(\frac{d}{dz}\right)^{N-1}\left[(z-z_0)^N f(z)\right]}{(N-1)!}\right|_{z=z_0}$$

3. Using the principal determination of log, calculate the residues of $\dfrac{\log z}{(z^2+1)^2}$ at each of its singularities.

4. For any nonnegative integer n, calculate the residues of $(z^2+1)^{-n-1}$ at each of its poles. Present your answer with the binomial coefficient $\binom{2n}{n}$ displayed prominently.

5. Find the residues of $\csc^n z$ at $z=0$ for $n=1,2,3,4,5$.

(That is, $\left(\dfrac{1}{\sin z}\right)^n$.)

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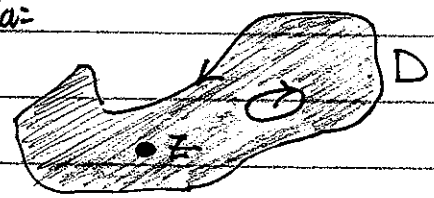
REMARK ABOUT THE RESIDUE THEOREM: It contains the Cauchy integral formula. (Of course, we actually used the Cauchy integral theorem in its proof.) For consider the usual scene for the Cauchy formula:



f holomorphic

We write the expected integral $\frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta$ and notice that for fixed $z \in D$ the function $f(\zeta)/\zeta - z$ is a holomorphic function of ζ with one isolated singularity, at z. And it's the easy case

$$\operatorname{Res}\left(\frac{f(\zeta)}{\zeta - z}, z\right) = \frac{f(z)}{\frac{d}{d\zeta}(\zeta - z)} = \frac{f(z)}{1} = f(z).$$

So indeed the residue theorem $\Rightarrow$

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z).$$

WHAT'S AHEAD FOR US: The residue theorem is an amazing tool for accomplishing all sorts of things in complex analysis. It can produce wonderful theoretical results and also astonishing computations. We could present these in either order. However, I prefer the computational aspects first because these techniques will give us lots of practice in dealing with our new concept of residues, and I think will also give us a nice change of pace in the middle of the semester.

So here we go!

EVALUATION OF SOME DEFINITE INTEGRALS

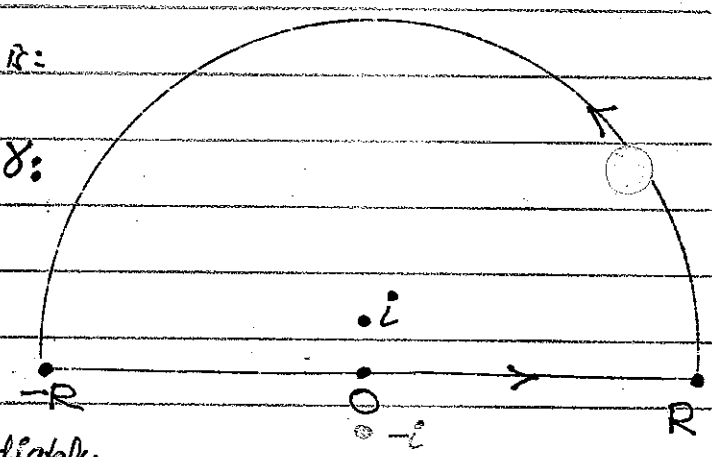
1. An example for babes

A quite elementary integral in basic calculus is $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi$. Now we approach it in an entirely different way, much more complicated than actually necessary for such a problem ... but the technique will lead the way for more interesting situations.

- Define the holomorphic function $f(z) = \frac{1}{z^2+1}$. This function has isolated singularities at i and $-i$.

- Devise a clever path. Here it is: for large R we hope to approximate the desired integral, and we have a pole of f inside.

γ:



The <u>residue theorem</u> gives immediately

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z^2+1}\,dz = \operatorname{Res}\left(\frac{1}{z^2+1}, i\right)$$
$$= \frac{1}{2z}\Big|_{z=i}$$
$$= \frac{1}{2i}.$$

Thus
$$\int_\gamma \frac{dz}{z^2+1} = \pi.$$

- Let $R \to \infty$. We have

$$\int_{-R}^{R} \frac{dx}{x^2+1} + \int_{\text{semicircle}} \frac{dz}{z^2+1} = \pi.$$

Feb 18(3)

The real integral is just what we want. We do not want to evaluate the semicircular integral but instead to show that it has limit 0 as $R \to \infty$. So we employ the basic estimate for path integrals:

(see Feb 6(2)) $$\left| \int_{\text{semicircle}} \frac{dz}{z^2+1} \right| \leq \max \left| \frac{1}{z^2+1} \right| \cdot \text{length of curve.}$$

The length of the curve is πR. And for $|z| = R$ we have from the triangle inequality

$$|z^2+1| \geq |z^2| - 1 = |z|^2 - 1 = R^2 - 1.$$

Thus we achieve the estimate

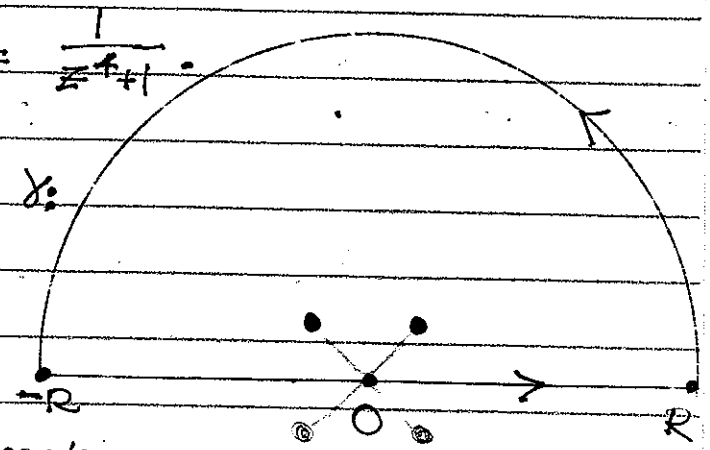
$$\frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty.$$

- Final result:

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^2+1} = \pi.$$

2. <u>A more challenging example</u>

Use the above <u>template</u> for $f(z) = \dfrac{1}{z^4+1}$.



The denominator has (simple) zeros at the fourth roots of $-1 = e^{\pi i}$:

$e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}$

And the residue of f at each one equal

$$\frac{1}{4z^3} = \frac{z}{4z^4} = -\frac{z}{4}.$$

The residue theorem $\Rightarrow$

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z^4+1} = \text{sum of residues at } e^{\frac{\pi i}{4}} \text{ and } e^{\frac{3\pi i}{4}}$$

$$= -\frac{e^{\pi i/4}}{4} - \frac{e^{3\pi i/4}}{4}$$

$$= -\frac{\frac{1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}}}{4}$$

$$= -\frac{i\sqrt{2}}{4}.$$

Thus
$$\int_\gamma \frac{dz}{z^4+1} = 2\pi i\left(\frac{-i\sqrt{2}}{4}\right) = \frac{\pi}{\sqrt{2}}.$$

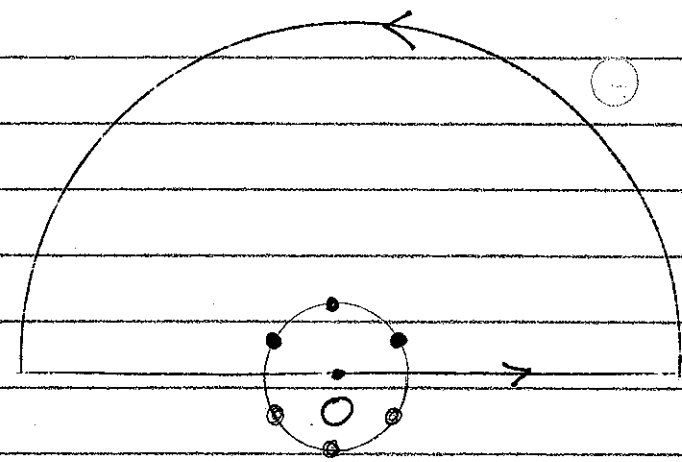
Again, the integral on the semicircle has modulus bounded by
$$\frac{\pi R}{R^4-1} \longrightarrow 0.$$

Conclusion:
$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}.}$$

3. <u>Another</u>

Use $f(z) = \frac{1}{z^6+1}$.

Six poles this time:
$$e^{\pi i/6},\ i,\ e^{5\pi i/6}, \text{etc.}$$



Everything works the same way.

The sum of the three residues is
$$\sum \frac{1}{6z^5} = \sum \frac{z}{6z^6} = -\frac{1}{6}\sum z$$
$$= -\frac{1}{6}\left[e^{\pi i/6} + i + e^{5\pi i/6}\right]$$
$$= -\frac{1}{6}\left[\frac{\sqrt{3}}{2} + \frac{i}{2} + i + \frac{-\sqrt{3}}{2} + \frac{i}{2}\right]$$
$$= -\frac{1}{6}\cdot 2i = -i/3.$$

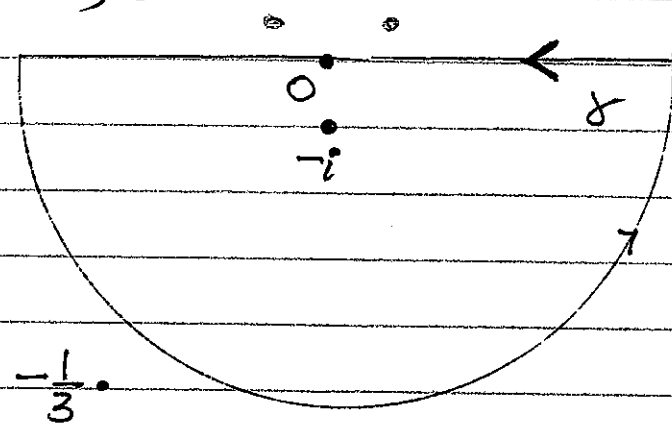
So the integral we obtain is $2\pi i(-i/3) = 2\pi/3$:
$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}.}$$

4. <u>Another</u>

Let's try $f(z) = \dfrac{1}{z^3 - i}$.

The three poles are roots of $z^3 = i = e^{\frac{\pi i}{2}} = e^{\frac{5\pi i}{2}} = e^{\frac{9\pi i}{2}}$,

so we obtain $z = e^{\pi i/6}, e^{5\pi i/6}, e^{\frac{3\pi i}{2}}$.



This time we save a small amount of work by using the <u>lower</u> semicircle, so we deal with one residue only. The residue at $-i$ equals $\dfrac{1}{3z^2} = \dfrac{1}{3(-i)^2} = -\dfrac{1}{3}$.

So the residue theorem gives $\displaystyle\int_\gamma \frac{1}{z^3 - i}\,dz = -\frac{2\pi i}{3}$.

Again the integral on the semicircle tends to 0 as $R \to \infty$, so our limiting equation is $\displaystyle\int_\infty^{-\infty} \frac{dx}{x^3 - i} = -\frac{2\pi i}{3}$.

Reverse the direction: $\boxed{\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^3 - i} = \frac{2\pi i}{3}.}$

(P.S. We could have solved it this way:

$$\int_{-\infty}^{\infty} \frac{dx}{x^3 - i} = \int_{-\infty}^{\infty} \frac{x^3 + i}{x^6 + 1}\,dx = 0 + i\int_{-\infty}^{\infty} \frac{dx}{x^6 + 1}$$

$$= \frac{2\pi i}{3}$$

from the preceding example.)

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5. Long detailed discussion of another example: $\int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx$, $\alpha \in \mathbb{R}$.

a. Convergence issues:

Near ∞ the integrand is approximately $x^{\alpha-2}$, so we require $\underline{\underline{\alpha<1}}$.
Near 0 the integrand is approximately $x^{\alpha-1}$, so we require $\underline{\underline{\alpha>0}}$.
Then, $\underline{\underline{0<\alpha<1}}$.

b. Choose a holomorphic function:

Let $f(z) = \frac{z^{\alpha-1}}{z+1}$, but we realize we'll have to cope with the ambiguity in

$$z^\alpha = \exp(\alpha \log z) = \exp(\alpha(\log|z| + i\arg z))$$
$$= |z|^\alpha \exp(i\alpha \arg z).$$

c. Residues:

$z=0$ is a <u>branch point</u>, not an isolated singularity. There is an isolated singularity at $z=-1$, and the residue is easy:

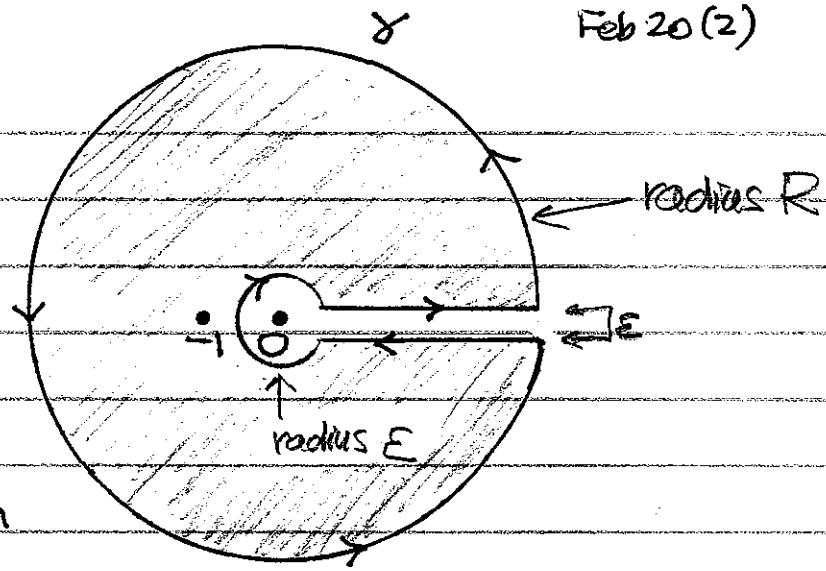
$$\operatorname{Res}(f,-1) = \frac{(-1)^{\alpha-1}}{1} = -(-1)^\alpha.$$
$$= -\exp(i\alpha \arg(-1)).$$

d. Path of integration:

Rather tricky! We need to have the positive real axis as part of the path, we need it to surround -1, and we need 0 to be outside!

Here is what we do:

γ is the boundary of the shaded region.



In this region we should use the choice of $\arg z$ so that $\underline{0 < \arg z < 2\pi}$.

Then we have

$$\mathrm{Res}(f,-1) = -\exp(i\alpha\pi) = -e^{i\alpha\pi}.$$

e. Residue theorem yields immediately that

$$\int_\gamma \frac{z^{\alpha-1}}{z+1}\,dz = -2\pi i\, e^{i\alpha\pi}.$$

f. $R \to \infty$

The integral on the circle is bounded by

$$\max_{|z|=R} |f(z)| \cdot 2\pi R = \max_{|z|=R} \frac{R^{\alpha-1}}{|z+1|} \cdot 2\pi R$$

$$\leq \frac{R^{\alpha-1}}{R-1} \cdot 2\pi R \overset{\text{approx}}{=} 2\pi R^{\alpha-1}.$$

Since $\underline{\alpha<1}$ this tends to 0 as $R \to \infty$.

g. $\varepsilon \to 0$

The integral on the circle is bounded by

$$\max_{|z|=\varepsilon} \frac{\varepsilon^{\alpha-1}}{|z+1|} \cdot 2\pi\varepsilon$$

$$\leq \frac{\varepsilon^{\alpha-1}}{1-\varepsilon} \cdot 2\pi\varepsilon \overset{\text{approx}}{=} 2\pi\varepsilon^{\alpha}.$$

Since $\underline{\alpha > 0}$ this tends to 0 as $\varepsilon \to 0$.

REMARK: our criteria for convergence of the derived integral match perfectly with what is needed in the path integral as $R \to \infty$, $\varepsilon \to 0$.

h. We have left two integrals along the positive real axis.
In the "upper" one $z = x$ with argument 0, so it becomes

$$\int_0^\infty \frac{z^{\alpha-1}}{z+1}\,dz = \int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx =: I.$$

But in the "lower" one $z = x$ with argument 2π, so it becomes

$$-\int_0^\infty \frac{z^{\alpha-1}}{z+1}\,dz = -\int_0^\infty \frac{x^{\alpha-1}e^{i\alpha 2\pi}}{x+1}\,dx = -e^{i\alpha 2\pi} I.$$

i. Summary: the equation in e. becomes in the limit

$$I - e^{i\alpha 2\pi} I = -2\pi i e^{i\alpha\pi}.$$

Solve for I:

$$I = \frac{-2\pi i e^{i\alpha\pi}}{1 - e^{i\alpha 2\pi}} = \frac{2\pi i}{e^{i\alpha\pi} - e^{-i\alpha\pi}}$$

$$= \frac{\pi}{\sin \alpha\pi}.$$

CONCLUSION:

$$\boxed{\int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx = \frac{\pi}{\sin\alpha\pi} \text{ for } 0 < \alpha < 1.}$$

This is just about the easiest example of this type of analysis, but I have gone to great detail to justify all the reasoning. After some practice this should become almost routine for you.

<u>REMARK</u>: #5 on HW 14F . . . to find the residues

$$\mathrm{Res}\left(\csc^n z, 0\right) \quad \text{for } n = 1, 2, \cdots.$$

The easy cases are

$n=1$: simple pole, residue $= 1$.

$n = 2, 4, 6, \cdots$ even function, residue $= 0$.

<u>$n=3$</u> Here's a beautiful and elegant technique, which I choose to call <u>integration by parts</u>. It's based on the fact that if f is a holomorphic function defined on a closed path γ, then

$$\int_\gamma f'(z)\,dz = 0.$$

This is a FTC fact, see Feb 1 (1). Now apply this to a product fg of holomorphic functions:

$$0 = \int_\gamma (fg)'\,dz = \int_\gamma f'g\,dz + \int_\gamma fg'\,dz,$$

or

$$\int_\gamma f'g\,dz = -\int_\gamma fg'\,dz. \quad \text{"INTEGRATION BY PARTS"}$$

In the case of functions with isolated singularities at z_0, when we use a small circle γ surrounding z_0 we obtain

$$\boxed{\mathrm{Res}(f'g, z_0) = -\mathrm{Res}(fg', z_0).}$$

Now we try this on $\csc^3 z$ near $z_0 = 0$. Then

$$\sin^{-3} z = \left(\frac{\sin^{-2} z}{-2}\right)' \sec z,$$

so

$$\operatorname{Res}(\sin^{-3} z, 0) = \operatorname{Res}\left(\frac{\sin^{-2} z}{} \sec' z, 0\right) \qquad \text{IBP}$$

$$= \operatorname{Res}\left(\frac{\sin^{-2} z}{2} \sec z \tan z, 0\right)$$

$$= \operatorname{Res}\left(\frac{\sin^{-1} z}{2} \sec^2 z, 0\right).$$

Look what just happened! We started with a pole of order 3, and now we have a pole of order 1 ! So we're in the <u>easy</u> case,

$$\operatorname{Res}\left(\frac{\sec^2 z}{2\sin z}, 0\right) = \frac{1}{2\cos 0} = \frac{1}{2}.$$

REMARK: "Integration by parts" is somewhat of a misnomer. For Definition **3** gives the result immediately that a derivative of a holomorphic function has zero residue, since on page Feb 15(2) we simply take $a = 0$. Nevertheless, I like the IBP name for this principle, as the result is so reminiscent of such a procedure.

Feb 20 (6)

HW 14F SOLUTIONS

1 For any positive integer n, find all the singularities of the function $\frac{1}{z^n+1}$ and calculate all the corresponding residues. Also, compute the sum of all the residues.

The singularities occur when $z^n = -1$. There are n distinct roots, which can be calculated this way:

$$z^n = -1 = e^{\pi i} = e^{3\pi i} = \cdots = e^{(2n-1)\pi i},$$

so the roots are

$$\boxed{e^{\frac{\pi i}{n}}, e^{\frac{3\pi i}{n}}, \ldots, e^{\frac{(2n-1)\pi i}{n}}}$$

If $z = e^{\frac{(2k+1)\pi i}{n}}$, then

$$\operatorname{Res}\left(\frac{1}{z^n+1}, z\right) = \frac{1}{nz^{n-1}} = \frac{z}{nz^n} = -\frac{z}{n}$$

$$= \boxed{-\frac{1}{n} e^{\frac{(2k+1)\pi i}{n}}.}$$

The sum of these residues is

$$-\frac{1}{n} e^{\frac{\pi i}{n}} \sum_{k=0}^{n-1} e^{\frac{2k\pi i}{n}},$$

which, if $n \geq$, equals

$$-\frac{1}{n} e^{\frac{\pi i}{n}} \frac{e^{\frac{2\pi i}{n} n} - 1}{e^{\frac{2\pi i}{n}} - 1} \leftarrow \text{geometric series} = 0.$$

Thus the sum equals

$$\boxed{\begin{cases} -1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}}$$

2 Suppose that f has a pole at z_0 of order $\leq N$. The function $(z-z_0)^N f(z)$ has a removable singularity at z_0. Prove that

$$\mathrm{Res}(f, z_0) = \frac{\left(\frac{d}{dz_0}\right)^{N-1}\left[(z-z_0)^N f(z)\right]}{(N-1)!}\Bigg|_{z=z_0}$$

Sufficient to do this for z_0. Then, the Laurent expansion of f is

$$f(z) = \sum_{n=-N}^{\infty} c_n z^n,$$

and

$$z^N f(z) = \sum_{n=-N}^{\infty} c_n z^{n+N}$$

$$= c_{-N} + c_{-N+1} z + \cdots + c_{-1} z^{N-1} + c_0 z^N + \cdots.$$

Then

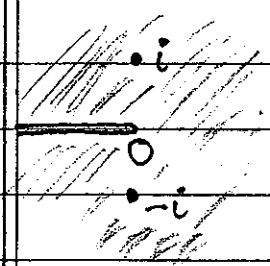
$$\left(\frac{d}{dz}\right)^{N-1} z^N f(z) = c_{-1}(N-1)! + \text{terms with } z^k \text{ for } k \geq 1.$$

Evaluate at 0 to get $c_{-1}(N-1)! = (N-1)!\,\mathrm{Res}(f, 0)$.

QED

3 Using the principal determination of log, calculate the residues of $\dfrac{\log z}{(z^2+1)^2}$ at each of its singularities.

Use the above result. As i and $-i$ are poles of order 2,



$$\text{Residue at } \pm i = \frac{d}{dz}\left[(z \mp i)^2 \frac{\log z}{(z^2+1)^2}\right]\Bigg|_{\pm i}$$

$$= \frac{d}{dz}\left[\frac{\log z}{(z \pm i)^2}\right]_{\pm i}$$

$$= \frac{\frac{1}{z}}{(z \pm i)^2} - \frac{2\log z}{(z \pm i)^3}\Bigg|_{\pm i} \qquad \text{(product rule)}$$

$$= \frac{\frac{1}{\pm i}}{(2(\pm i))^2} - \frac{2\log \pm i}{(2(\pm i))^3}$$

$$= \frac{\mp i}{4i^2} - \frac{2 \pm \frac{i\pi}{2}}{8(\pm i^3)}$$

$$= \frac{\pm i}{4} \mp \frac{i\pi}{8(\mp i)} = \frac{\pi}{8} \pm \frac{i}{4}$$

4 For any nonnegative integer n, calculate the residues of $(z^2+1)^{-n-1}$ at each of its poles. Present your answer with the binomial coefficient $\binom{2n}{n}$ displayed prominently.

Very similar result. Pole of order $n+1$ at i, so we form

$$\left(\frac{z-i}{z^2+1}\right)^{n+1} = \left(\frac{1}{z+i}\right)^{n+1} = (z+i)^{-n-1};$$

calculate the nth derivative:

$$(-n-1)(-n-2)\cdots(-2n)\,(z+i)^{-2n-1}$$
$$= (-1)^n (2n)(2n-1)\cdots(n+1)\,(z+i)^{-2n-1}.$$

evaluate at i:

$$(-1)^n (2n)\cdots(n+1)\, 2^{-2n-1}\, i^{-2n-1}$$
$$= (2n)\cdots(n+1)\, 2^{-2n-1}\, i^{-1};$$

divide by $n!$:

$$\binom{2n}{n} 2^{-2n-1} i^{-1}.$$

$$\text{Residue at } i = \frac{-\binom{2n}{n} i}{2^{2n+1}}.$$

Likewise,

$$\text{Residue at } -i = \frac{\binom{2n}{n} i}{2^{2n+1}}.$$

There are several approaches to the next problem, and I want you to see one you might not think of. I think it's the best approach in this particular case. It's based on Definition **3** on page Feb 15(2). Here it is:

> If f has an isolated singularity at z_0, then
>
> $$\operatorname{Res}(f', z_0) = 0.$$

As a corollary, if f and g have isolated singularities at z_0, then the residue of $(fg)'$ at z_0 is 0: since this is $f'g + fg'$, we obtain a sort of "integration by parts" result:

$$\boxed{\operatorname{Res}(f'g, z_0) = -\operatorname{Res}(fg', z_0).}$$

5 Find the residues of $\csc^n z$ at $z=0$ for $n=1,2,3,4,5$.
(That is, $\left(\frac{1}{\sin z}\right)^n$.)

Immediately, $n=1$ gives residue 1, all even n give residue 0.

- $\csc^3 z = \sin^{-3} z = \left(\frac{-\sin^{-2} z}{2}\right)' \sec z$ and this has the same residue at 0 as $\frac{\sin^{-2} z}{2}(\sec z)' = \frac{\sin^{-2} z}{2} \sec z \tan z = \frac{\sin^{-1} z}{2} \sec^2 z$, and this has a <u>simple pole</u> at 0. Thus the residue equals

$$\operatorname{Res}\left(\frac{\sec^2 z}{2\sin z}, 0\right) = \frac{1}{2\cos 0} = \frac{1}{2}.$$

- $\csc^5 z = \sin^{-5} z = \left(\frac{-\sin^{-4} z}{4}\right)' \sec z$ has same residue as

$$\frac{\sin^{-4} z}{4} \sec z \tan z = \frac{\sin^{-3} z}{4} \sec^2 z$$
$$= \left(-\frac{\sin^{-2} z}{8}\right)' \sec^3 z;$$

same residue as

$$\frac{\sin^{-2} z}{8} (\sec^3 z)' = \frac{\sin^{-2} z}{8} 3 \sec^3 z \tan z$$
$$= \frac{3}{8} \sin^{-1} z \sec^4 z \ldots \text{ simple pole!}$$

The desired residue is $\therefore$ $\frac{3/8}{1} = \frac{3}{8}$.

MATH 382 Lecture notes for 2-22-13

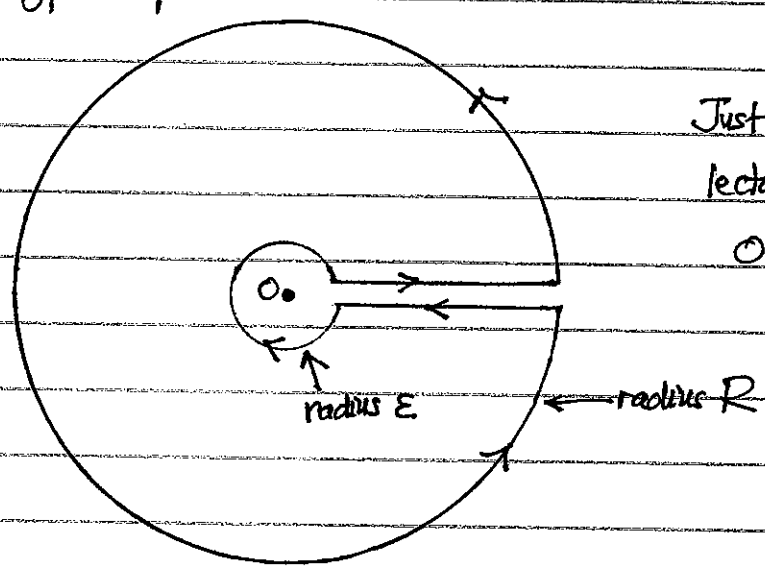
6. Integrals of a certain form:

$$\int_0^\infty \frac{a(x)}{b(x)}\,dx \quad \text{where:}$$

- a, b are polynomials
- degree $b \geq$ degree $a + 2$
- $b \neq 0$ on $[0,\infty) \subset \mathbb{R}$

Such situations can always be handled with residue theory, so we'll derive a general result and then apply it to a particular example.

The approach is quite clever! We define $\boxed{f(z) = \frac{a(z)}{b(z)} \log z}$ and use this type of path:



Just as in the preceding lecture, we use $0 < \arg z < 2\pi$.

The integrals on the two circles tend to 0 in the limit as $\varepsilon \to 0$ and $R \to \infty$. The extra factor of $\log z$ is of very little concern, since

$$|\log z| = |\log|z| + i\arg z| \leq |\log|z| + 2\pi i|.$$

So when $|z| = R$ we have $|\log z| \leq |\log R + 2\pi i| \leq 2\log R$

and for $|z| = \varepsilon$ $\quad |\log z| \leq |\log\varepsilon + 2\pi i| \leq 2\log\frac{1}{\varepsilon}$

as $R \to \infty$ and $\varepsilon \to 0$. Thus on the circle $|z| = R$ we have

$$\left| \int_{|z|=R} f(z)\,dz \right| \le \frac{\text{constant}}{R^2} \cdot \log R \cdot 2\pi R \le \frac{\text{constant} \cdot \log R}{R},$$

which tends to 0 as $R \to \infty$ ... for $\log R \to \infty$ much slower than R.
And on the circle $|z| = \varepsilon$ we have a similar estimate

$$\left| \int_{|z|=\varepsilon} f(z)\,dz \right| \le \text{constant} \cdot \log \tfrac{1}{\varepsilon} \cdot 2\pi\varepsilon \le \text{constant} \cdot \varepsilon \log \tfrac{1}{\varepsilon}.$$

Again this tends to 0 as $\varepsilon \to 0$.

So we apply the residue theorem and then let $R \to \infty$, $\varepsilon \to 0$, and obtain in the limit

$$\int_0^\infty \frac{a(x)}{b(x)} \log x \, dx - \int_0^\infty \frac{a(x)}{b(x)} \left[\log x + 2\pi i \right] dx$$

$$= 2\pi i \text{ times sum of the residues of } f(z).$$

Notice that on the path $(0,\infty)$ <u>above</u> this path $\log z = \log x$ ($\arg z = 0$), but on the path $(\infty, 0)$ <u>below</u> the axis, $\log z = \log |z| + 2\pi i$, because $\arg z = 2\pi$.

So when we subtract the integrals, $\log x$ disappears, and we're left with

$$-2\pi i \int_0^\infty \frac{a(x)}{b(x)}\,dx.$$

Divide by $-2\pi i$ to acheive the formula

$$\boxed{\int_0^\infty \frac{a(x)}{b(x)}\,dx = -\text{the sum of all residues of } \frac{a(z)}{b(z)} \log z.}$$

(Here $0 < \arg z < 2\pi$.)
(Though it doesn't matter!)

EXAMPLE: Let $a>0$ and consider

$$I = \int_0^\infty \frac{x}{(x+a)(x^2+1)}\,dx.$$

All our requirements are met. The poles are located at $-a$, i, and $-i$, and they are all simple! ☺ When the residues are computed we obtain

$$\operatorname{Res}\left(\frac{z}{(z+a)(z^2+1)}\log z, -a\right) = \frac{-a\log(-a)}{1\cdot(a^2+1)}$$

$$= \frac{-a(\log a + \pi i)}{a^2+1};$$

at i we have the residue

$$\frac{i\log i}{(i+a)2i} = \frac{i\left(\frac{i\pi}{2}\right)}{(a+i)2i} = \frac{i\pi}{4(a+i)} = \frac{i\pi(a-i)}{4(a^2+1)}.$$

And at $-i$

$$\frac{-i\log(-i)}{(-i+a)2(-i)} = \frac{-i\left(\frac{3i\pi}{2}\right)}{(a-i)(-2i)} = \frac{3i\pi}{4(a-i)} = \frac{3i\pi(a+i)}{4(a^2+1)}.$$

$$\frac{1}{a^2+1}\left\{\left(-a\log a - a\pi i\right) + \frac{i\pi a + \pi}{4} + \frac{3i\pi a - 3\pi}{4}\right\}$$

$$= \frac{1}{a^2+1}\left\{-a\log a - \frac{\pi}{2}\right\}.$$

Therefore we obtain from the formula the result that for $a>0$

$$\boxed{\int_0^\infty \frac{x}{(x+a)(x^2+1)}\,dx = \frac{a\log a + \frac{\pi}{2}}{a^2+1}.}$$

REMARK: Many examples we demonstrate can actually be done with single variable calculus. This is a good example, as are 1,2,3,4. We could even find the indefinite integral first. But even so, these techniques are exceedingly beautiful even in such cases!

HW 15W due Mar 6, 2013

For any integer $n \geq 2$, compute the integral

$$\int_0^\infty \frac{dx}{x^n + x^{n-1} + \cdots + x + 1}.$$

DISCUSSION: This fits what we have just done, with $a(z) = 1$ and $b(z) = z^n + z^{n-1} + \cdots + z + 1$. Here's an approach using lots of calculations:

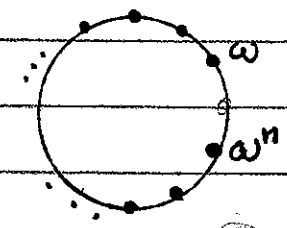
- Poles:

$$b(z) = \frac{z^{n+1} - 1}{z - 1}$$ so poles occur when $z^{n+1} = 1$ (and $z \neq 1$).

I suggest defining

$$\omega = e^{\frac{2\pi i}{n+1}},$$

so that the poles are ω^k for $1 \leq k \leq n$.



- Residues of $\dfrac{\log z}{b(z)}$ are happily

$$\frac{\log \omega^k}{b'(\omega^k)} = \frac{k \log \omega}{b'(\omega^k)} = \frac{2\pi i}{n+1} \frac{k}{b'(\omega^k)}.$$

For $z = \omega^k$ we have

$$b'(z) = -\frac{z^{n+1} - 1}{(z-1)^2} + \frac{(n+1)z^n}{z - 1} = \frac{n+1}{z^2 - z} \quad (\text{WHY?}).$$

- Formula of page 2 $\Longrightarrow$ our integral equals

$$-\sum_{k=1}^{n} \frac{2\pi i k}{n+1} \frac{\omega^{2k} - \omega^k}{n+1} = \frac{2\pi i}{(n+1)^2} \left(\sum_{k=1}^{n} k\omega^k - \sum_{k=1}^{n} k\omega^{2k} \right).$$

- Algebra: SHOW THAT the term ( ) on the preceding line equals

$$z\, b'(z) \Big|_{\omega^2}^{\omega} = (n+1)\left(\frac{\omega}{\omega^2 - \omega} - \frac{\omega^2}{\omega^4 - \omega^2} \right).$$

- Finish: Greatly simplify to get the final answer in the form

$$\frac{\pi}{(n+1)\sin(??)}.$$

MATH 382 Lecture notes for 3-4-13

7. <u>Principal value integrals</u> refer to reasonable attempts to define a sort of integral even when the integrand is not actually integrable. There are two different situations where this may occur. Here are <u>illustrations</u> of these types:

<u>Type 1</u> $\int_{-1}^{e} \frac{1}{x}\,dx$ doesn't actually exist, since the two "sub integrals" do not exist:

$$\int_{-1}^{0} \frac{1}{x}\,dx = -\infty \quad \text{and} \quad \int_{0}^{e} \frac{1}{x}\,dx = +\infty.$$

So what we may do is first delete a <u>symmetric</u> interval about 0 and then perform a limit:

$$\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{e} \frac{dx}{x} = \log|x|\Big|_{-1}^{-\varepsilon} + \log x\Big|_{\varepsilon}^{e}$$

$$= (\log\varepsilon - 0) + (1 - \log\varepsilon)$$

$$= 1.$$

So the limit as $\varepsilon \to 0$ does exist, and it called the <u>principal value integral</u>. Notation:

$$\mathrm{PV}\int_{-1}^{e} \frac{dx}{x} = 1.$$

<u>Type 2</u> $\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx$ again does not exist, for

$$\int_{-\infty}^{0} \frac{x}{x^2+1}\,dx = -\infty \quad \text{and} \quad \int_{0}^{\infty} \frac{x}{x^2+1}\,dx = +\infty.$$

But we can define a principal value by integrating from $-R$ to R and then letting $R \to \infty$. Notation:

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx = 0.$$

The residue theorem can often be of use in calculating such integrals. I'll give as an example a typical situation, and we'll see others.

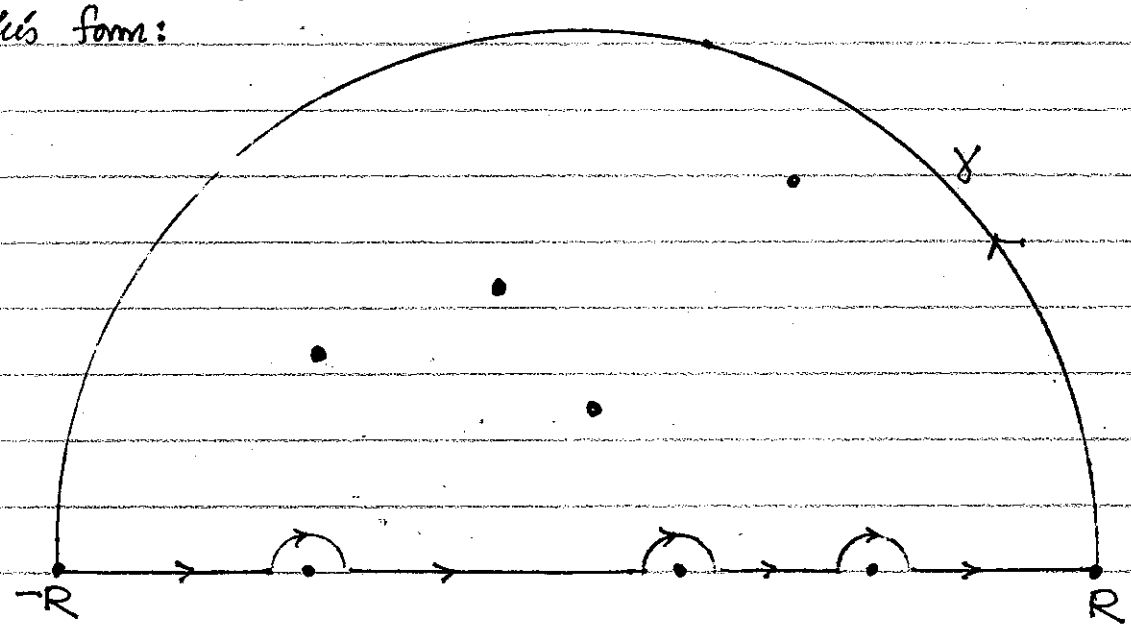
So assume $f(z)$ is a rational function of z for which

$$\underline{\text{degree of denominator} \geq 2 + \text{degree of numerator}},$$

just as on page Feb 22 (1).

Furthermore, if f has any <u>real</u> poles, assume they are <u>simple</u>.

We shall then apply the residue theorem when our path integral has this form:



We are of course familiar with the large semicircle. The new twist is that we have semicircles of small radius centered at the <u>real</u> (simple) poles of f.

R is so large that all the real poles of f are between $-R$ and R, and all the poles with imaginary part greater than 0 satisfy $|z| < R$.

The residue theorem then gives immediately the equation

$$\int_\gamma f(z)\,dz = 2\pi i \mathcal{R},$$

where we have denoted

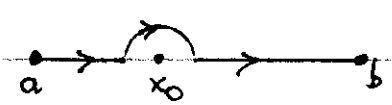
$$\mathcal{R} = \sum_{\mathrm{Im}(z)>0} \mathrm{Res}(f, z).$$

Because of the restriction on degrees of denominator and numerator of f, we can let $R \to \infty$ and obtain

$$\int_{\gamma_\varepsilon} f(z)\,dz = 2\pi i \mathcal{R},$$

where γ_ε represents the real axis with semicircular arcs of radius ε situated about the real poles of f.

It is fascinating to see what happens when we let $\varepsilon \to 0$. We can deal with each real pole individually. So look at a pole at $x_0 \in \mathbb{R}$ and the portion of γ_ε from a to b, when $a < x_0 < b$:



Parametrize the semicircle as $z = x_0 + \varepsilon e^{i\theta}$, where θ travels from π to 0. We obtain

$$\int_a^{x_0-\varepsilon} f(x)\,dx + \int_\pi^0 f(x_0+\varepsilon e^{i\theta})\, i\varepsilon e^{i\theta}\,d\theta + \int_{x_0+\varepsilon}^b f(x)\,dx.$$

The first and third of these integrals will have the limit as $\varepsilon \to 0$:

$$\mathrm{PV}\int_a^b f(x)\,dx \ \ldots \ \underline{\text{provided the limit exists}}!$$

This limit does indeed exist, as we see from analyzing the second integral:

from page Feb 15(4) we have

$$\mathrm{Res}(f, x_0) = \lim_{z \to x_0} (z - x_0) f(z) \qquad ☺$$

since x_0 is a simple pole. Thus

$$\mathrm{Res}(f, x_0) = \lim_{\varepsilon \to 0} \varepsilon e^{i\theta} f(x_0 + \varepsilon e^{i\theta}) \qquad (\underline{\text{uniformly}} \text{ w.r.t. } \theta)$$

and we therefore obtain

$$\lim_{\varepsilon \to 0} \int_{\pi}^{0} f(x_0 + \varepsilon e^{i\theta}) i \varepsilon e^{i\theta} d\theta = \int_{\pi}^{0} i \,\mathrm{Res}(f, x_0) d\theta$$
$$= -\pi i \,\mathrm{Res}\, f(x_0).$$

Doing this for each pole thus yields the formula

$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z) dz = PV \int_{-\infty}^{\infty} f(x) dx - \pi i \, \mathcal{R}^*,$$

where

$$\mathcal{R}^* = \sum_{x \in \mathbb{R}} \mathrm{Res}(f, x).$$

Thus we have derived the result, that under the given restrictions on f,

$$\boxed{PV \int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum_{\mathrm{Im}(z) > 0} \mathrm{Res}(f, z) + \pi i \sum_{x \in \mathbb{R}} \mathrm{Res}(f, x).}$$

EXAMPLE: $f(z) = \frac{1}{z^3+1}$. The real pole is at -1, and the residue there is $\frac{1}{3z^2} = \frac{1}{3}$. The other pole to consider is $e^{i\pi/3}$, with residue $\frac{1}{3z^2} = \frac{1}{3} e^{-\frac{2i\pi}{3}} = \frac{1}{3}\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$. Thus we have to calculate $2\pi i \left(-\frac{1}{6} - \frac{i}{2\sqrt{3}}\right) + \frac{\pi i}{3} = \frac{\pi}{\sqrt{3}}$, so we have

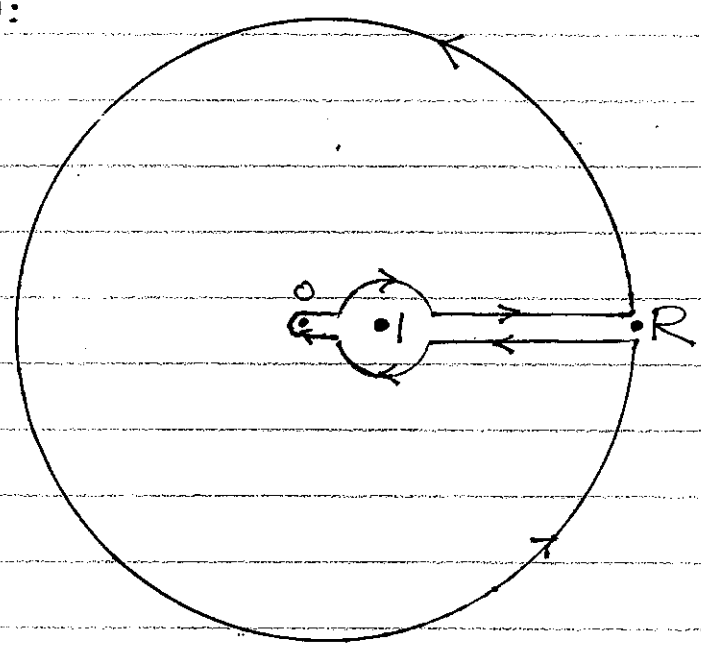
$$\boxed{PV\int_{-\infty}^{\infty} \frac{dx}{x^3+1} = \frac{\pi}{\sqrt{3}}.}$$

HW 16F due Mar 8, 2013

For $0<\alpha<1$ calculate

$$PV\int_0^{\infty} \frac{x^{\alpha-1}}{1-x}\,dx.$$

RECOMMENDATION:



$$f(z) = \frac{z^{\alpha-1}}{1-z}$$

with

$0 < \arg z < 2\pi.$

HW 15W SOLUTION From page Feb 22(4) we have

- Poles: $\omega = e^{\frac{2\pi i}{n+1}}$; poles ω^k for $1 \le k \le n$.

- Residues: give $b'(z) = -\dfrac{z^{n+1}-1}{(z-1)^2} + \dfrac{(n+1)z^n}{z-1} = \dfrac{(n+1)\overbrace{z^{n+1}}^{=1}}{z^2-z}$.

$\Rightarrow 0$ because $z^{n+1}=1$

- Formula of page Feb 22(2) gives the integral equal to

$$-\sum_{k=1}^{n} \operatorname{Res}\left(\frac{\log z}{b(z)}, \omega^k\right) = -\sum_{k=1}^{n} \frac{2\pi i k}{(n+1)\, b'(\omega^k)}$$

$$= -\sum_{k=1}^{n} \frac{2\pi i k}{n+1} \frac{\omega^{2k}-\omega^k}{n+1}$$

$$= \frac{2\pi i}{(n+1)^2}\left(\sum_{k=1}^{n} k\omega^k - \sum_{k=1}^{n} k\omega^{2k}\right).$$

But

$$b(z) = \sum_{k=0}^{n} z^k \Rightarrow z b'(z) = \sum_{k=1}^{n} k z^k.$$

Thus our integral equals

$$\frac{2\pi i}{(n+1)^2}\left(\omega b'(\omega) - \omega^2 b'(\omega^2)\right) = \frac{2\pi i}{n+1}\left(\frac{\omega}{\omega^2-\omega} - \frac{\omega^2}{\omega^4-\omega^2}\right).$$

Finish: This is

$$\frac{2\pi i}{n+1}\left(\frac{1}{\omega-1} - \frac{1}{\omega^2-1}\right) = \frac{2\pi i}{n+1}\,\frac{(\omega+1)-1}{\omega^2-1} = \frac{2\pi i}{n+1}\,\frac{\omega}{\omega^2-1}$$

$$= \frac{2\pi i}{n+1}\,\frac{1}{\omega-\frac{1}{\omega}} = \frac{2\pi i}{n+1}\,\frac{1}{e^{\frac{2\pi i}{n+1}} - e^{-\frac{2\pi i}{n+1}}}$$

$$= \frac{\pi}{n+1}\,\frac{1}{\sin\frac{2\pi}{n+1}}.$$

$$\therefore \quad \boxed{\int_0^\infty \frac{dx}{x^n+x^{n-1}+\cdots+x+1} = \frac{\pi}{(n+1)\sin\frac{2\pi}{n+1}}.}$$

MATH 382 Lecture notes for 3-6-13

8. Integrals involving exponentials (and trig functions)

This is an extremely important type of definite integrals, and we'll gain extra proficiency in our techniques as well as in our use of exp. Today we'll look at two rather typical examples.

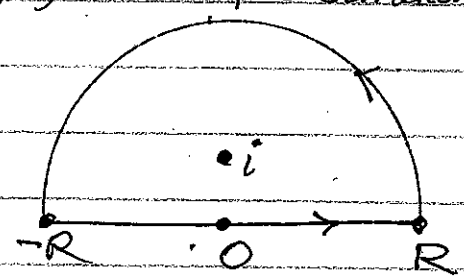
Example 1 $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx$.

First, recall the simple equation $|e^w| = e^{\operatorname{Re} w}$. Therefore,

$$|e^{iax}| = e^{\operatorname{Re}(iax)} = e^{-\operatorname{Im}(ax)} = e^{-x\operatorname{Im}(a)}.$$

So if $\operatorname{Im}(a) \neq 0$, the integrand blows up exponentially as $x \to \infty$ or as $x \to -\infty$, and we have no existing integral. Therefore, we definitely must assume that $a \in \mathbb{R}$.

We are led to define $f(z) = \dfrac{e^{iaz}}{z^2+1}$, a holomorphic function with isolated simple poles at $\pm i$.



Let's try our familiar semicircle: knowing we'll need to worry about the path $|z| = R$, we examine

$$|f(z)| = \frac{|e^{iaz}|}{|z^2+1|} = \frac{e^{-\operatorname{Im}(az)}}{|z^2+1|}$$

$$\leq \frac{e^{-a\operatorname{Im}(z)}}{|z^2|-1} = \frac{e^{-a\operatorname{Im}(z)}}{R^2-1}.$$

We'll be in bad trouble if $a<0$! (Since $\operatorname{Im}(z)>0$.) Therefore, we also assume $a \geq 0$. Then we have for $|z|=R$

$$|f(z)| \leq \frac{1}{R^2-1},$$

and we conclude that since the length of the semicircle is πR, the path integral

$$\left| \int_{\substack{|z|=R \\ \mathrm{Im}(z)>0}} f(z)\,dz \right| \leq \frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty.$$

Thus the residue theorem implies (after letting $R \to \infty$) that

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \,\mathrm{Res}(f(z), i)$$

$$\overset{\text{☺}}{=} 2\pi i \,\frac{e^{iai}}{2i} = \pi e^{-a}.$$

Thus we obtain

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a} \text{ for } a \geq 0.$$

Finish: if $a<0$ we obtain immediately by conjugation the result πe^{a}. Therefore we have in general

$$\boxed{\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-|a|} \text{ for all } a \in \mathbb{R}.}$$

What a terrific result! If $a=0$ this is a very elementary integral since arctan has derivative $\frac{1}{x^2+1}$. But for $a \neq 0$ there's no convenient indefinite integral.

<u>Example 2</u> $\displaystyle\int_{-\infty}^{\infty} \frac{x e^{iax}}{x^2+1}\,dx.$

Again we must assume that $a \in \mathbb{R}$. We'll also first deal with the case $a>0$. (The case $a=0$ is quite different, as

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx$$

exists only in the principal value sense --- and clearly is 0.)

Thus we define
$$f(z) = \frac{z e^{iaz}}{z^2+1}.$$

The residue at $z=i$ is $\dfrac{i e^{iai}}{2i} = \dfrac{1}{2} e^{-a}$ ☺.

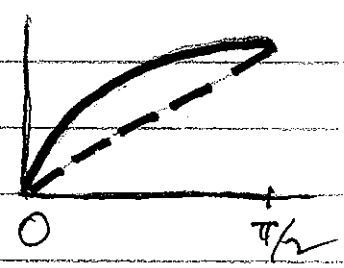
We employ the same semicircular path, and we first try to estimate the path integral along $|z|=R$:

$$\left| \int_{\substack{|z|=R \\ \operatorname{Im} z>0}} f(z)\,dz \right| \le \int_{\substack{|z|=R \\ \operatorname{Im} z>0}} \frac{|z| e^{-a \operatorname{Im} z}}{|z|^2-1} |dz|$$
$$= \frac{R}{R^2-1} \int_0^{\pi} e^{-aR\sin\theta} R\,d\theta$$
$$= \frac{R^2}{R^2-1} \int_0^{\pi} e^{-aR\sin\theta}\,d\theta.$$

Uh oh! We can no longer simply use the estimate $e^{-aR\sin\theta} \le 1$, so we have to be cleverer. Not knowing how to integrate $e^{-aR\sin\theta}$, we employ a useful estimate. First, we can integrate from 0 to $\frac{\pi}{2}$ only and double the answer to get the estimate

$$\frac{2R^2}{R^2-1} \int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta < 3\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}\,d\theta$$

for large R. Then we estimate $\sin\theta$ <u>from</u> <u>below</u> by observing its graph:



$\sin\theta \ge \frac{2\theta}{\pi}$ for $0 \le \theta \le \frac{\pi}{2}$.

Therefore we find
$$\int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta < \int_0^{\frac{\pi}{2}} e^{-\frac{aR2\theta}{\pi}}\,d\theta$$
$$< \int_0^{\infty} e^{-\frac{aR2\theta}{\pi}}\,d\theta = \frac{\pi}{2aR}!$$

Conclusion: the path integral of $f(z)$ along the semicircle has modulus no bigger than $\frac{R^2}{R^2-1}\frac{3\pi}{2aR} \to 0$ as $R \to \infty$.

Thus we again obtain from the residue theorem

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \operatorname{Res}(f(z), i)$$
$$= 2\pi i \frac{e^{-a}}{2}$$
$$= \pi i e^{-a}.$$

Here is the result for all $a \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+1}dx = \begin{cases} \pi i e^{-a} & \text{for } a>0, \\ 0 & \text{for } a=0 \text{ (PV integral)}, \\ -\pi i e^{a} & \text{for } a<0. \end{cases}$$

SUMMARY:

Using Euler's formula $e^{iax} = \cos ax + i \sin ax$, we see that the symmetry of the integrands gives the two results in this form:

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}dx = \pi e^{-|a|} \text{ for } a \in \mathbb{R};$$

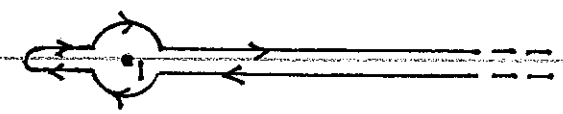
$$\int_{-\infty}^{\infty} \frac{x \sin ax}{x^2+1}dx = \begin{cases} \pi e^{-a} & \text{for } a>0, \\ 0 & \text{for } a=0, \\ -\pi e^{a} & \text{for } a<0. \end{cases}$$

REMARK: The integrals $\int_{-\infty}^{\infty} \frac{x \sin ax}{x^2+1}dx$ are <u>not</u> principal value integrals, as the integrand is an even function of x. However, they are <u>improper</u> integrals as they are not absolutely integrable:

$$\int_{-\infty}^{\infty} \left|\frac{x \sin ax}{x^2+1}\right| dx = \infty \quad (\text{for } a \neq 0).$$

HW 16F SOLUTION Let $f(z) = \dfrac{z^{\alpha-1}}{1-z}$ with $0 < \arg z < 2\pi$.

1. The path integral around a small circle centered at 0 tends to 0 as the radius tends to 0 because $\alpha > 0$.
2. The path integral around a large circle centered at 0 tends to 0 as the radius tends to ∞ because $\alpha < 1$.
3. The remaining figure is this:



4. As f has no poles inside the region, the residue theorem yields the remaining path integral $= 0$. Along the top part of the path we have

$$\int_0^{1-\varepsilon} \frac{x^{\alpha-1}}{1-x}\,dx + \int_{1+\varepsilon}^{\infty} \frac{x^{\alpha-1}}{1-x}\,dx + \int_{\curvearrowright} \frac{z^{\alpha-1}}{1-z}\,dz \qquad (\arg z \geq 0)$$

and along the bottom part

$$\int_{\infty}^{1+\varepsilon} \frac{x^{\alpha-1} e^{2\pi i(\alpha-1)}}{1-x}\,dx + \int_{1-\varepsilon}^{0} \frac{x^{\alpha-1} e^{2\pi i(\alpha-1)}}{1-x}\,dx + \int_{\curvearrowleft} \frac{z^{\alpha-1}}{1-z}\,dz \qquad (\arg z \leq 2\pi).$$

5. Top part as $\varepsilon \to 0$ has limit

$$PV\int_0^{\infty} \frac{x^{\alpha-1}}{1-x}\,dx - \pi i \operatorname{Res}\left(\frac{z^{\alpha-1}}{1-z}, 1\right)$$

and the bottom part has limit

$$PV\int_{\infty}^{0} \frac{x^{\alpha-1}}{1-x}\,dx\; e^{2\pi i\alpha} - \pi i \operatorname{Res}\left(\frac{z^{\alpha-1}}{1-z}, 1\right).$$

6. Thus our situation is now that this sum is 0:

$$PV\int_0^{\infty} \frac{x^{\alpha-1}}{1-x}\,dx \left[1 - e^{2\pi i\alpha}\right] = \underbrace{\pi i \operatorname{Res}\left(\frac{z^{\alpha-1}}{1-z}, 1\right)}_{\text{TOP}} + \underbrace{\pi i \operatorname{Res}\left(\frac{z^{\alpha-1}}{1-z}, 1\right)}_{\text{BOTTOM}} = 0$$

7. These residues ☺ are different since at the point 1 "on the top"

$$1^{\alpha-1} = 1,$$

whereas "on the bottom"

$$1^{\alpha-1} = e^{2\pi i(\alpha-1)} = e^{2\pi i\alpha}.$$

Therefore the sum on the right side is

$$\pi i\left[-1 - e^{2\pi i\alpha}\right].$$

8. Now solve for the desired integral:

$$PV\int_0^\infty \frac{x^{\alpha-1}}{1-x}\,dx = \pi i\,\frac{e^{2\pi i\alpha}+1}{e^{2\pi i\alpha}-1}$$

$$= \pi i\,\frac{e^{\pi i\alpha}+e^{-\pi i\alpha}}{e^{\pi i\alpha}-e^{-\pi i\alpha}}$$

$$= \pi i\,\frac{2\cos\pi\alpha}{2i\sin\pi\alpha}.$$

9. Final result:

$$\boxed{PV\int_0^\infty \frac{x^{\alpha-1}}{1-x}\,dx = \pi\cot\pi\alpha.}$$

MATH 382

EXAM 2 Due March 11, 2013

1. Suppose that f and g are entire holomorphic functions such that

$$|f(z)| \leq |g(z)| \text{ for all } z \in \mathbb{C}.$$

Prove that there exists a constant c such that

$$f(z) = c\,g(z) \text{ for all } z \in \mathbb{C}.$$

2. The tangent function is of course defined by

$$\tan z = \frac{\sin z}{\cos z}.$$

a. Denote the Maclaurin series of $\tan z$ by

$$\tan z = \sum_{n=0}^{\infty} c_n z^n.$$

What is the radius of convergence of this series?

b. Prove that $c_n = 0$ for all even n.

c. Prove that $c_n > 0$ for all odd n.

3. Let $a(z)$ and $b(z)$ be polynomials such that

$$\text{degree of } b \geq 2 + \text{degree of } a.$$

Find the sum of all the residues of the function $\dfrac{a(z)}{b(z)}$.

4. For any complex number a calculate

$$\int_{-\infty}^{\infty} \frac{1}{(x-a)(x-i)}\,dx$$

(if $a \in \mathbb{R}$ this is a principal value integral).

For # 4 give me your <u>answer</u> only

HONOR PLEDGE

MATH 382 Lecture notes for 3-8-13

Two more examples involving trigonometric functions

- The first can be found in almost every textbook on complex analysis. It's the integral

$$\int_0^\infty \frac{\sin ax}{x}\,dx \quad , \text{ where as usual } a \in \mathbb{R}.$$

This integral is of course 0 if $a=0$. Otherwise, it is an <u>improper</u> integral, since

$$\int_0^\infty \left|\frac{\sin ax}{x}\right| dx = \infty. \quad \text{(Not hard to show.)}$$

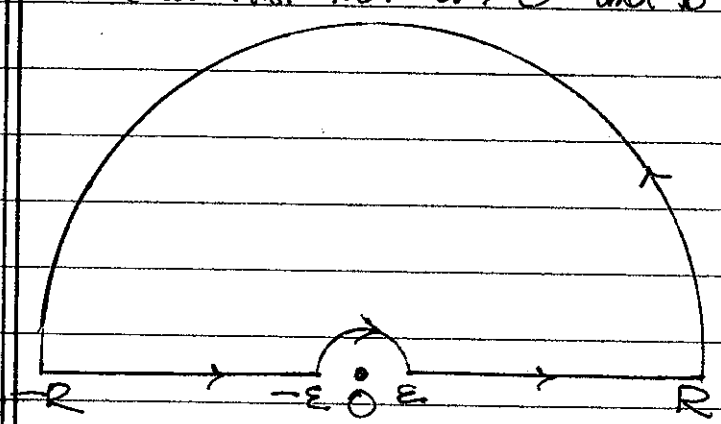
Thus it has to be interpreted as

$$\lim_{R\to\infty} \int_0^R \frac{\sin ax}{x}\,dx .$$

The choice of holomorphic function is crucial! We must <u>not</u> choose $\frac{\sin az}{z}$ because of its large modulus when $\operatorname{Im} z \neq 0$. Therefore, the logical choice is

$$f(z) = \frac{e^{iaz}}{z}.$$

(This actually introduces a <u>pole</u> at 0.) Prior experience leads us to assume at first that $a>0$ and to choose a path like this:



Our earlier estimates on page Mar 6 (3) show that the integral along $|z|=R$ tends to 0, <u>since $a>0$</u>. Thus the residue theorem gives

$$\int_{-\infty}^{-\varepsilon} f(x)\,dx + \int_{\cap} f(z)\,dz + \int_{\varepsilon}^{\infty} f(x)\,dx = 0.$$

Then our work on March 4 yields in the limit as $\varepsilon \to 0$

$$PV\int_{-\infty}^{\infty} f(x)\,dx - \pi i \operatorname{Res}(f,0) = 0.$$

This ☺ residue is 1, so our result is

$$PV\int_{-\infty}^{\infty} \frac{e^{iax}}{x}\,dx = \pi i \quad \text{for } a>0.$$

i.e.

$$PV\int_{-\infty}^{\infty} \frac{\cos ax + i\sin ax}{x}\,dx = \pi i.$$

All that survives from this equation is

$$\int_{-\infty}^{\infty} \frac{\sin ax}{x}\,dx = \pi \quad \text{for } a>0.$$

Note: no PV is left, as that goes only with the cosine term. Or we could have taken the imaginary part of each side.

Since the integrand is <u>even</u> as a function of x, we obtain

$$\int_{0}^{\infty} \frac{\sin ax}{x}\,dx = \frac{\pi}{2} \quad \text{for } a>0.$$

Finally, since $\sin ax$ is an <u>odd</u> function of a, our final result is

$$\boxed{\int_{0}^{\infty} \frac{\sin ax}{x}\,dx = \begin{cases} \frac{\pi}{2} & \text{if } a>0, \\ 0 & \text{if } a=0, \\ -\frac{\pi}{2} & \text{if } a<0. \end{cases}}$$

- Our second example is the integral

$$\int_{0}^{\infty} \frac{1-\cos ax}{x^2}\,dx.$$

(This is a <u>proper</u> integral, thanks to the boundedness of the integrand as $x \to 0$ and its decay at ∞ like x^{-2}.)

Our experience leads us to assume $a>0$ and to choose

$$f(z) = \frac{1-e^{iaz}}{z^2}.$$

This function has a simple pole at $z=0$, with

$$\operatorname{Res}(f,0) = \operatorname{Res}\left(\frac{-iaz+\cdots}{z^2}, 0\right)$$
$$= -ia.$$

Using the same path as above and letting $R\to\infty$, $\varepsilon\to 0$, we obtain

$$PV\int_{-\infty}^{\infty} \frac{1-e^{iax}}{x^2}\,dx = \pi i(-ia) = \pi a.$$

And then Euler's formula gives

$$\int_{-\infty}^{\infty} \frac{1-\cos ax}{x^2}\,dx = \pi a \text{ for } a>0.$$

Final result:

$$\boxed{\int_0^{\infty} \frac{1-\cos ax}{x^2}\,dx = \frac{\pi|a|}{2} \text{ for all } a\in\mathbb{R}.}$$

(A standard trig identity $\Rightarrow$

$$\int_0^{\infty} \frac{2\sin^2\frac{ax}{2}}{x^2}\,dx = \frac{\pi|a|}{2}\ )$$

and then $(a\to 2a)$

$$\boxed{\int_0^{\infty} \frac{\sin^2 ax}{x^2}\,dx = \frac{\pi|a|}{2}.}\ )$$

HW 17W due Mar 13, 2013

Calculate

$$PV\int_0^{\infty} \frac{\cos ax}{x^2-1}\,dx \quad \text{for } a\in\mathbb{R}.$$

SOLUTIONS FOR EXAM 2

Mar 8 (4)

1. Suppose that f and g are entire holomorphic functions such that
$$|f(z)| \le |g(z)| \quad \text{for all } z \in \mathbb{C}.$$
Prove that there exists a constant c such that
$$f(z) = c\,g(z) \quad \text{for all } z \in \mathbb{C}.$$

Trivial case: $g(z) = 0$ for all z. ✓

Otherwise, the zeros of g are isolated. Define $h(z) := \dfrac{f(z)}{g(z)}$ at all points where $g(z) \neq 0$. Then h is holomorphic and $|h(z)| \le 1$. The zeros of g are isolated singularities of h, and the fact that h is bounded implies these singularities are <u>removable</u> (Riemann's theorem). Remove those singularities so that h becomes an entire holomorphic function. Then Liouville's theorem $\Rightarrow$ h is constant (since it's bounded). Call the constant c. Then $f/g = c$ so $f = cg$. **QED**

2. The tangent function is of course defined by
$$\tan z = \frac{\sin z}{\cos z}.$$

a. Denote the Maclaurin series of $\tan z$ by
$$\tan z = \sum_{n=0}^{\infty} c_n z^n.$$
What is the radius of convergence of this series?

The function tan is holomorphic except for (simple) poles at the zeros of cos. We know these to be $\frac{\pi}{2} + \mathbb{Z}\pi$. Thus the poles of tan closest to 0 are $\pm\frac{\pi}{2}$.

a. $R = \frac{\pi}{2}$.

b. Prove that $c_n = 0$ for all even n.
c. Prove that $c_n > 0$ for all odd n.

b. As $\tan(-z) = -\tan z$, all even terms $c_n = 0$.

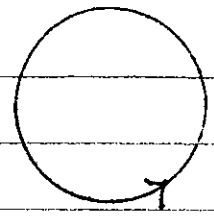
c. Repeat the argument on page Feb 6 (6) or else <u>use that result</u>: $\tan' = \sec^2 = \left(\sum_0^\infty a_n z^{2n}\right)^2$, all $a_n > 0$.

Do the squaring: $\tan' = \sum_0^\infty b_n z^{2n}$, all $b_n > 0$.

Integrate: $\tan z = \sum_0^\infty \frac{b_n}{2n+1} z^{2n+1}$, so $c_n = \frac{b_n}{2n+1} > 0$.

QED

3. Let $a(z)$ and $b(z)$ be polynomials such that degree of $b \geq 2 +$ degree of a. Find the sum of all the residues of the function $\frac{a(z)}{b(z)}$.



For sufficiently large R, all the poles of $\frac{a(z)}{b(z)}$ satisfy $|z|<R$. The residue theorem implies that

$$\int_{|z|=R} \frac{a(z)}{b(z)}\,dz = 2\pi i \sum \text{residues}.$$

But for sufficiently large R we have $\left|\frac{a(z)}{b(z)}\right| \leq \frac{\text{const}}{R^2}$. Thus

$$2\pi\left|\sum \text{residues}\right| = \left|\int_{|z|=R} \frac{a(z)}{b(z)}\,dz\right|$$

$$\leq \frac{\text{const}}{R^2}\cdot 2\pi R \to 0 \text{ as } R\to\infty.$$

Since LHS does not depend on R, it must be 0. QED

4. For any complex number a calculate

$$\int_{-\infty}^{\infty} \frac{1}{(x-a)(x-i)}\,dx$$

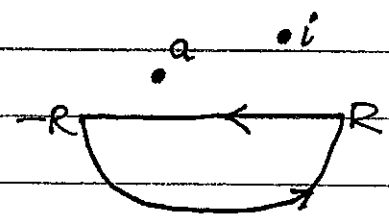
(if $a \in \mathbb{R}$ this is a principal value integral).

For #4 give me your answer only

$$f(z) = \frac{1}{(z-a)(z-i)}$$

Case 1 $\operatorname{Im} a > 0$

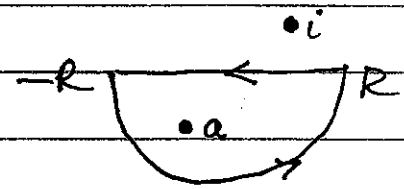
Let $R \to \infty$ to get



$$\int_{\infty}^{-\infty} f(x)\,dx = 0.$$

So the answer is 0

Case 2 $\underline{\operatorname{Im} a < 0}$

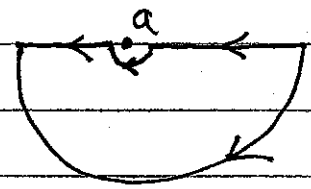


We get

$$\int_{\infty}^{-\infty} f(x)\,dx = 2\pi i \operatorname{Res}(f,a)$$

$$\overset{\text{☺}}{=} \frac{2\pi i}{a-i}, \text{ so the answer is } \frac{2\pi i}{i-a}$$

Case 3 $\underline{\operatorname{Im} a = 0}$



We get

$$PV\int_{\infty}^{-\infty} f(x)\,dx = \pi i \operatorname{Res}(f,a) = \frac{\pi i}{a-i}, \text{ so the answer is}$$

$$\frac{\pi i}{i-a}$$

SUMMARY:

$$\int_{-\infty}^{\infty} \frac{dx}{(x-a)(x-i)} = \begin{cases} 0 & \text{if } \operatorname{Im} a > 0 \\ \dfrac{\pi i}{i-a} & \text{if } \operatorname{Im} a = 0 \\ \dfrac{2\pi i}{i-a} & \text{if } \operatorname{Im} a < 0 \end{cases}$$