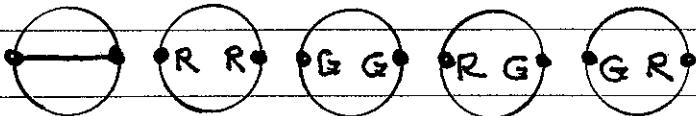


EXAM 2 (Catalan Exam): due Wednesday March 19

1. Given n points on a circle, how many ways are there to connect some pairs of them with disjoint chords and to assign the color red or the color green to the rest of the n points.

Example for $n=2$:



(5 ways)

Of course, you must provide a proof.

2. Prove this Catalan identity: $C_{n+1} = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} C_k$.

(You may use the first problem.)

3. Prove that the Motzkin number M_n is equal to

$$M_n = \sum_{k \geq 0} \binom{n}{2k} C_k.$$

4. Here is the situation we discussed Monday: define the doubly infinite sequence $a_{n,k}$ by

$$\begin{cases} a_{n,k} = 0 & \text{if } k < 0 \text{ or } k > n, \\ a_{n,k} = a_{n-1,k-1} + a_{n-1,k+1} & \text{for } n \geq 1, \\ a_{0,0} = 1. \end{cases}$$

I think of the infinite lower triangular matrix

$A = (a_{n,k})$ where $\begin{cases} n = \text{row number} \\ k = \text{column number} \end{cases}$:

$$A = \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 2 & 0 & 3 & 0 & 1 & 0 & \cdots \\ 0 & 5 & 0 & 4 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right)$$

Find a formula for the 0^{th} column: $a_{n,0}$ for all $n \geq 0$.
And prove your result.

(I suggest that you begin by analyzing the generating functions
for the columns =

$$A_k(z) = \sum_{n \geq 0} a_{n,k} z^n.$$

(I learned this sort of thing in a 1999 paper by Martin Aigner.)

END OF EXAM 2

Amusing exercise from Richard P. Stanley's Enumerative Combinatorics Vol 2:

- 6.24. [?] Explain the significance of the following sequence:

un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, ...