MATH 468 Handout # 10 March 12, 2014

EXAM 2 (Catalan Exam): due Wednesday March 19

1. Given n points on a circle, how many ways are there to connect some pairs of them with disjoint chords and to assign the color <u>red</u> or the color <u>green</u> to the rest of the n points.

Example for $n=2$: 

(5 ways)

Of course, you must provide a proof.

2. Prove this Catalan identity: $C_{n+1} = \sum_{k\geq 0} \binom{n}{2k} 2^{n-2k} C_k$.

(You may use the first problem.)

3. Prove that the Motzkin number M_n is equal to

$$M_n = \sum_{k\geq 0} \binom{n}{2k} C_k .$$

4. Here is the situation we discussed Monday: define the doubly infinite sequence $a_{n,k}$ by

$$\begin{cases} a_{n,k} = 0 \text{ if } k<0 \text{ or } k>n, \\ a_{n,k} = a_{n-1,k-1} + a_{n-1,k+1} \text{ for } n\geq 1, \\ a_{0,0} = 1. \end{cases}$$

I think of the infinite lower triangular matrix

$$A = (a_{n,k}) \quad \text{where} \begin{cases} n = \text{row number} \\ k = \text{column number} \end{cases}:$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 2 & 0 & 3 & 0 & 1 & 0 & \cdots \\ 0 & 5 & 0 & 4 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Find a formula for the 0th column: $a_{n,0}$ for all $n \geq 0$.
And prove your result.

(I suggest that you begin by analyzing the generating functions for the columns:

$$A_k(z) = \sum_{n \geq 0} a_{n,k} z^n.)$$

(I learned this sort of thing in a 1999 paper by Martin Aigner.)

END OF EXAM 2

Amusing exercise from Richard P. Stanley's Enumerative Combinatorics Vol 2:

6.24. [?] Explain the significance of the following sequence:

un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, . . .