

Remarks concerning Problem 4 of Exam 2, based on the paper of Martin Aigner, *Catalan-like Numbers and Determinants*, Journal of Combinatorial Theory, Series A **87**, 33-51 (1999)

What motivated this study was to understand the following characterization of the Catalan numbers:

Hankel matrix

The $n \times n$ Hankel matrix whose (i, j) entry is the Catalan number C_{i+j-2} has determinant 1, regardless of the value of n . For example, for $n = 4$ we have

$$\det \begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 14 \\ 2 & 5 & 14 & 42 \\ 5 & 14 & 42 & 132 \end{bmatrix} = 1.$$

Moreover, if the indexing is "shifted" so that the (i, j) entry is filled with the Catalan number C_{i+j-1} then the determinant is still 1, regardless of the value of n . For example, for $n = 4$ we have

$$\det \begin{bmatrix} 1 & 2 & 5 & 14 \\ 2 & 5 & 14 & 42 \\ 5 & 14 & 42 & 132 \\ 14 & 42 & 132 & 429 \end{bmatrix} = 1.$$

Taken together, these two conditions uniquely define the Catalan numbers.

In Exam 2 we considered the doubly infinite sequence $a_{n,k}$:

$$\begin{cases} a_{n,k} = 0 & \text{if } k < 0 \text{ or } k > n, \\ a_{n,k} = a_{n-1,k-1} + a_{n-1,k+1} & \text{for } n \geq 1, \\ a_{0,0} = 1. \end{cases}$$

We then proved that

$$a_{2n,0} = \text{the } n\text{th Catalan number } C_n.$$

(Of course, $a_{n,k} = 0$ if $n+k$ is odd.)

Aigner's paper has a fascinating property concerning the rows

$R_n := (a_{n,0} \ a_{n,1} \ a_{n,2} \ a_{n,3} \ \dots)$ of these numbers. It is

thus: the dot product of two rows is given by

$$R_m \cdot R_n = a_{m+n,0}.$$

This is a rather easy induction on m .

BASE CASE $R_0 \cdot R_n = a_{n,0}$ for all n .

For $R_0 = (1 \ 0 \ 0 \ 0 \ \dots)$ so $R_0 \cdot R_n = a_{n,0}$.

INDUCTIVE STEP Suppose $R_m \cdot R_n = 0$ for all n , and then calculate $R_{m+1} \cdot R_n$. This is pretty easy and pretty:

$$\begin{aligned} R_{m+1} \cdot R_n &= \sum_k a_{m+1,k} a_{n,k} \\ &= \sum_k (a_{m,k-1} + a_{m,k+1}) a_{n,k} \\ &= \sum_k a_{m,k-1} a_{n,k} + \sum_k a_{m,k+1} a_{n,k} \\ &= \sum_k a_{m,k} a_{n,k+1} + \sum_k a_{m,k} a_{n,k-1} \\ &= \sum_k a_{m,k} (a_{n,k+1} + a_{n,k-1}) \\ &= \sum_k a_{m,k} a_{n+1,k} \\ &= R_m \cdot R_{n+1} \\ &= a_{m+(n+1),0} \quad (\text{inductive hypothesis}) \\ &= a_{(m+1)+n,0}. \quad \text{QED} \end{aligned}$$

Now consider any NW corner submatrix, using rows $0, 1, \dots, N$:

$$A = \begin{pmatrix} R_0 \\ R_1 \\ \vdots \\ R_N \end{pmatrix}.$$

Now set

$$H_k = \det \begin{pmatrix} C_0 & \dots & C_k \\ \vdots & & \vdots \\ C_k & \dots & C_{2k} \end{pmatrix}, \text{ for } k \geq 0,$$

and

$$J_k = \det \begin{pmatrix} C_1 & \dots & C_k \\ \vdots & & \vdots \\ C_k & \dots & C_{2k-1} \end{pmatrix}, \text{ for } k \geq 1.$$

Then we conclude that

$$H_{\frac{N-1}{2}} J_{\frac{N+1}{2}} = 1 \text{ for odd } N,$$

$$H_{\frac{N}{2}} J_{\frac{N}{2}} = 1 \text{ for even } N.$$

Now $H_0 = 1$, so $J_1 = 1$ (using $N=1$);

$J_1 = 1$, so $H_1 = 1$ (using $N=2$);

$H_1 = 1$, so $J_2 = 1$ (using $N=3$);

$J_2 = 1$, so $H_2 = 1$ (using $N=4$); etc.

Conclusion:

$$H_k = 1 \text{ and } J_k = 1 \text{ for all } k.$$

QED

Somebody please help me: one solution for this problem in Exam 2 was the discovery that for $n+k$ even

$$a_{n,k} = \frac{k+1}{\frac{n+k}{2}+1} \binom{n}{\frac{n+k}{2}}.$$

It's easy enough to check that this is valid.

PLEASE DERIVE THIS FORMULA!