

Notes on the Stirling asymptotic expansion of the gamma function

Stirling asymptotic expansion of the gamma function

Start from Euler summation on p. 469. Use $a=0$, $b=n$, and $f(x) = \ln(x+\alpha)$, where $\alpha > 0$ is fixed. Then

$$\begin{aligned} \int_0^n f(x) dx &\stackrel{\text{IBP}}{=} (x+\alpha) \ln(x+\alpha) - x \Big|_0^n \\ &= (n+\alpha) \ln(n+\alpha) - \alpha \ln \alpha - n; \\ f' &= \frac{1}{x+\alpha}, \quad f'' = \frac{-1}{(x+\alpha)^2}, \quad \dots, \quad f^{(k)} = \frac{(-1)^{k-1} (k-1)!}{(x+\alpha)^k}. \end{aligned}$$

Then we obtain for any $m \geq 2$

$$\begin{aligned} \sum_{k=0}^{n-1} \ln(k+\alpha) &= (n+\alpha) \ln(n+\alpha) - \alpha \ln \alpha - n + B_1(f'(n) - f'(0)) \\ &\quad + \sum_{k=2}^m \frac{B_k}{k!} \frac{(-1)^k (k-2)!}{(x+\alpha)^{k-1}} \Big|_0^n + R_m, \end{aligned}$$

where

$$R_m = (-1)^{m-1} \int_0^n \frac{B_m(\{x\})}{m!} \frac{(-1)^{m-1} (m-1)!}{(x+\alpha)^m} dx.$$

Now insert $B_1 = -\frac{1}{2}$ and with an eye to letting $n \rightarrow \infty$ when the time comes,

$$\begin{aligned} \sum_{k=0}^{n-1} \ln(k+\alpha) &= (n+\alpha - \frac{1}{2}) \ln(n+\alpha) - (\alpha - \frac{1}{2}) \ln \alpha - n \\ &\quad - \sum_{k=2}^m \frac{B_k}{k(k-1)} \frac{1}{\alpha^{k-1}} + O\left(\frac{1}{n}\right) + R_m^0, \end{aligned}$$

where

$$R_m^0 = \int_0^\infty \frac{B_m(\{x\})}{m! (x+\alpha)^m} dx.$$

Further,

$$\begin{aligned}\ln(n+x) &= \ln n + \ln\left(1 + \frac{x}{n}\right) \\ &= \ln n + \frac{x}{n} + O\left(\frac{1}{n^2}\right),\end{aligned}$$

so

$$(n+\alpha-1) \ln(n+\alpha) = (n+\alpha-\frac{1}{2}) \ln n + \alpha + O\left(\frac{1}{n}\right)$$

We have this formula for the gamma function:

$$\begin{aligned}\Gamma(\alpha) &= \lim_{n \rightarrow \infty} \frac{n! n^\alpha}{\alpha(\alpha+1)\dots(\alpha+n)} \\ &= \lim_{n \rightarrow \infty} \frac{(n-1)! n^\alpha}{\alpha(\alpha+1)\dots(\alpha+n-1)} \quad (\text{easy adjustment}).\end{aligned}$$

Thus

$$\begin{aligned}\ln \Gamma(\alpha) &= \lim_{n \rightarrow \infty} \left(\ln(n-1)! + \alpha \ln n - \sum_{k=0}^{n-1} \ln(k+\alpha) \right) \\ &= \lim_{n \rightarrow \infty} \left(\ln(n-1)! + \alpha \ln n - (n-\frac{1}{2}) \ln n - \alpha + O\left(\frac{1}{n}\right) + \right. \\ &\quad \left. + (\alpha-\frac{1}{2}) \ln \alpha + \sum_{k=2}^m \frac{B_k}{k(k-1)} \frac{1}{\alpha^{k-1}} - R_m^\circ \right) \\ &= (\alpha-\frac{1}{2}) \ln \alpha - \alpha + \sum_{k=2}^m \frac{B_k}{k(k-1)} \frac{1}{\alpha^{k-1}} - R_m^\circ + \sigma,\end{aligned}$$

where $\sigma = \lim_{n \rightarrow \infty} (\ln(n-1)! - (n-\frac{1}{2}) \ln n + n)$. Not clear by itself that this limit even exists, except that it figures in an equation which guarantees its existence! Thus we see that

$$\begin{aligned}\ln \Gamma(\alpha) &= (\alpha-\frac{1}{2}) \ln \alpha - \alpha + \sigma + \sum_{k=2}^m \frac{B_k}{k(k-1)} \frac{1}{\alpha^{k-1}} \\ &\quad - \int_0^\infty \frac{B_m(-x)}{m} \frac{dx}{(x+\alpha)^m}.\end{aligned}$$