DOCUMENTATION FOR *KNOTTWISTER*

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Abstract. We summarize the mathematics behind the computer program *KnotTwister* and explain some of its features.

Contents

1. Definitions

1.1. Alexander polynomials. Let M be a 3–manifold and $\phi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z})$. Let $\mathbb{F}$ be a field and let $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F},k)$ be a presentation. Then we can consider $\mathbb{F}^k[t^{\pm 1}] := \mathbb{F}^k \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}]$ as a left $\mathbb{Z}[\pi_1(M)]$–module as follows:

$$g \cdot v := t^{\phi(g)}\alpha(g)(v),$$

where $g \in \pi_1(M), v \in \mathbb{F}^k[t^{\pm 1}]$. Now denote by $\tilde{M}$ the universal cover of M. Then the chain groups $C_*(\tilde{M})$ are in a natural way right $\mathbb{Z}[\pi_1(M)]$–modules. We can therefore form the tensor product $C_*(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{F}^k[t^{\pm 1}]$. Now define $H_*^\alpha(M,\mathbb{F}^k[t^{\pm 1}]) := H_*(C_*(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{F}^k[t^{\pm 1}])$.

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Note that $H_i^\alpha(M, \mathbb{F}^k[t^{\pm 1}])$ is a module over the principal ideal domain (PID henceforth) $\mathbb{F}[t^{\pm 1}]$. Therefore there exists an isomorphism

$$H_i^\alpha(M, \mathbb{F}^k[t^{\pm 1}]) \cong \mathbb{F}[t^{\pm 1}]^f \oplus \bigoplus_{i=1}^k \mathbb{F}[t^{\pm 1}]/(p_i(t))$$

for $p_1(t), \dots, p_k(t) \in \mathbb{F}[t^{\pm 1}]$. We define

$$\Delta_i^\alpha := \begin{cases} \prod_{i=1}^k p_i(t), & \text{if } f = 0 \\ 0, & \text{if } f > 0 \end{cases}$$

This is called the *twisted Alexander polynomial* of (M, ϕ, α). We furthermore define $\tilde{\Delta}_i^\alpha := \prod_{i=1}^k p_i(t)$ regardless of f.

It follows from the structure theorem of modules over PID's that these polynomials are well–defined up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$. In Section 4 we will see that $\Delta_i^\alpha(t)$ and $\tilde{\Delta}_i^\alpha(t)$ can be computed easily for $i = 0, 1$ given a presentation of $\pi_1(M)$.

Remark. Twisted Alexander polynomials for knots were defined by Lin [Lin01]. This was generalized by Jiang and Wang [JW93] and multivariable twisted Alexander polynomials over a UFD were first introduced by Wada [Wa94]. These definitions differ slightly from our definition (cf. [KL99a]).

For an oriented knot K we always assume that ϕ denotes the generator of $H^1(X(K); \mathbb{Z})$ given by the orientation. If $\alpha : \pi_1(X(K)) \to \mathrm{GL}(\mathbb{Q}, 1)$ is the trivial representation the Alexander polynomial then $\Delta_1^\alpha(t)$ equals the classical Alexander polynomial $\Delta_K(t)$ of a knot.

An important source of finite representations are homomorphisms to finite groups. Indeed, let $\alpha : \pi_1(M) \to G$ be a homomorphism to a finite group. Then G acts by left multiplication on the group ring $\mathbb{F}[G]$ which as a vector space is isomorphic to $\mathbb{F}^{|G|}$. In particular we get a representation $\pi_1(M) \to \mathrm{GL}(\mathbb{F}, |G|)$. We denote the resulting Alexander polynomial by $\Delta^G_{\phi,M}(t)$, suppressing the homomorphism α in the notation.

1.2. **The Thurston norm.** For a connected CW complex X denote by $\chi(X)$ the Euler characteristic, and we define $\chi_-(X) := \max(-\chi(X), 0)$. In general define $\chi_-(X) = \sum \chi_-(X_i)$ where we sum over the connected components of X. This is called the *complexity* of X.

Let M be a 3–manifold. The *Thurston norm* of $\phi \in H^1(M; \mathbb{Z})$ is defined as

$$||\phi||_T := \min\{\chi_-(S)\},$$

where we take the minimum with respect to all properly embedded surfaces S dual to ϕ.

The most important example are knot complements. Let $K \subset S^3$ be a knot, denote by $N(K)$ an open neighborhood and let $X(K) := S^3 \setminus N(K)$. Then $||\phi||_T = 2\,\mathrm{genus}(K) - 1$.

2. Lower bounds on the Thurston norm and fibered manifolds

For $f=\sum_{i=m}^{n} a_i t^i \in \mathbb{F}[t^{\pm 1}]$ with $a_m \neq 0, a_n \neq 0$ we define $\deg(f)=n-m$. Note that $\deg(\Delta_1^\alpha(t))$ is well–defined.

Theorem 2.1. [FK05] *Let M be a 3–manifold which is either closed or whose boundary consists only of tori. Let $\phi \in H^1(M;\mathbb{Z})$ be non–trivial and $\alpha : \pi_1(M) \to GL(\mathbb{F},k)$ a representation such that $\Delta_1^\alpha(t) \neq 0$. Then*

$$||\phi||_T \geq \frac{1}{k}\big(\deg(\Delta_1^\alpha(t)) - \deg(\Delta_0^\alpha(t)) - \deg(\Delta_2^\alpha(t))\big).$$

If α is unitary, then

$$||\phi||_T \geq \frac{1}{k}\big(\deg(\Delta_1^\alpha(t)) - (1+b_3(M))\deg(\Delta_0^\alpha(t))\big).$$

In Section 4 we will show how to compute $\Delta_1^\alpha(t)$ and $\Delta_0^\alpha(t)$. Since the Thurston norm is an integer any estimate can always be rounded up to the next integer. In many cases one can in fact easily determine whether $||\phi||_T$ is even or odd (cf. [FK05]).

We consider a few special cases:

(1) The trivial representation: McMullen's theorem,
(2) Abelian representations: Turaev's theorem,
(3) representations to group rings.

Theorem 2.2. [FK05] [Mc02, Proposition 6.1] *Let M be a 3–manifold and $\phi \in H^1(M;\mathbb{Z})$ primitive. If $\Delta_1(t) \neq 0$, then for any field $\mathbb{F}$*

$$||\phi||_T \geq \deg(\Delta_1(t)) - 1 - \tilde{b}_3(M).$$

Theorem 2.3. [FK05] [Tu02b] *Let M be a 3–manifold, $\phi \in H^1(M;\mathbb{Z})$ primitive, and $\alpha : \pi_1(M) \to H_1(M,\mathbb{Z}) \to GL(\mathbb{F},1)$ a one–dimensional representation which is non–trivial on Ker(ϕ). If $\Delta_1^\alpha(t) \neq 0$, then*

$$||\phi||_T \geq \deg(\Delta_1^\alpha(t)).$$

Theorem 2.4. [FK05] *Let M be a 3–manifold, $\phi \in H^1(M;\mathbb{Z})$ primitive, and $\alpha : \pi_1(M) \to G$ an epimomorphism to a finite group. If $\Delta_1^G(t) \neq 0$ then*

$$||\phi||_T \geq \frac{1}{|G|}\big(\deg(\Delta_1^G(t)) - n(\phi,\alpha)(1+\tilde{b}_3(M))\big)$$

where $n(\phi,\alpha) \in \mathbb{N}$ is the divisibility of $\phi|_{ker(\alpha)}$.

Note that $n(\phi,\alpha)$ is easy to compute since Ker(α) is finitely generated and is computed by *KnotTwister*.

Let L be a boundary link (for example a split link). It is well–known that the multivariable Alexander polynomial of L vanishes (cf. [Hi02]). With a little extra care it is not hard to show that the twisted multivariable and twisted one–variable

Alexander polynomials vanish as well. Therefore Theorem 2.1 can not be applied to get lower bounds on the Thurston norm. But for links we still have the following result.

Theorem 2.5. [FK05] *Let $L = L_1 \cup \dots \cup L_k$ be a link, denote its meridians by $\mu_1, \dots, \mu_k$. Let $\phi \in H^1(X(L);\mathbb{Z})$ primitive and dual to a meridian, i.e. $\phi(\mu_i) = 1$ for some i and $\phi(\mu_j) = 0$ for $j \neq i$. Then*

$$||\phi||_T \geq \frac{1}{k}\deg(\tilde{\Delta}_1^\alpha(t)) - 1.$$

Let M be a 3–manifold and $\phi \in H^1(M;\mathbb{Z})$. We say (M,ϕ) fibers over S^1 if the homotopy class of maps $M \to S^1$ induced by $\phi : \pi_1(M) \to H_1(M,\mathbb{Z}) \to \mathbb{Z}$ contains a representative that is a fiber bundle over S^1. If K is a fibered knot, i.e. if $X(K)$ fibers, then it is a classical result that genus$(K) = \frac{1}{2}\deg(\Delta_K(t))$ and that $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic, i.e. its top coefficient is $+1$ or -1.

Theorem 2.6. [FK05] *Let M be a 3–manifold and $\phi : \pi_1(M) \to \mathbb{Z}$ surjective, such that (M,ϕ) fibers over S^1 and such that $M \neq S^1 \times D^2, M \neq S^1 \times S^2$. Let $\alpha : \pi_1(M) \to GL(\mathbb{F},k)$ be a representation. Then $\Delta_1^\alpha(t) \neq 0$ and Then*

$$||\phi||_T = \frac{1}{k}\big(\deg\left(\Delta_1^\alpha(t)\right) - \deg\left(\Delta_0^\alpha(t)\right) - \deg\left(\Delta_2^\alpha(t)\right)\big).$$

If α is unitary, then

$$||\phi||_T = \frac{1}{k}\big(\deg\left(\Delta_1^\alpha(t)\right) - (1+b_3(M))\deg\left(\Delta_0^\alpha(t)\right)\big).$$

Since $||\phi||_T$ might be unknown for a given example the following corollary gives a more practical fibration obstruction.

Corollary 2.7. [FK05] *Let M be a 3–manifold and $\phi : \pi_1(M) \to \mathbb{Z}$ surjective such that (M,ϕ) fibers over S^1 and such that $M \neq S^1 \times D^2, M \neq S^1 \times S^2$. Let $\mathbb{F}, \mathbb{F}'$ be fields. Consider $\Delta_1(t) \in \mathbb{F}[t^{\pm 1}]$. For a representation $\alpha : \pi_1(M) \to GL(\mathbb{F}',k)$ we have*

$$\deg(\Delta_1(t)) - 1 - b_3(M) = \frac{1}{k}\big(\deg\left(\Delta_1^\alpha(t)\right) - \deg\left(\Delta_0^\alpha(t)\right) - \deg\left(\Delta_2^\alpha(t)\right)\big).$$

This result clearly generalizes the first condition on fibered knots. But it also contains the second condition. Indeed, if $X(K)$ is fibered then it follows from Theorem 2.6 that for different prime numbers p the degrees of $\Delta_K(t) \in \mathbb{F}_p[t^{\pm 1}]$ are the same. Clearly this implies that $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic. In [FK05] we show that this fibering obstruction contains Cha's fibering obstructions [Ch03].

3. Twisted Alexander polynomials as isotopy invariants

The twisted Alexander polynomial of a knot depends on the choice of a representation. In particular twisted Alexander polynomials are a priori not isotopy invariants.

In this section we propose a simple approach to obtaining isotopy invariants which turn out to be very strong and practical knot invariants.

In the following let $R=\mathbb{Z}$ or $R=\mathbb{F}_p$ a finite field. Given a map $\alpha:\pi_1(X)\to S_k$ we get an induced representation $\pi_1(X)\to S_k\to \mathrm{GL}(R,k)$ where S_k acts on R^k by permuting the coordinates. We denote this representation by $\overline{\alpha}$.

Now let K be a knot. Define $R_k(K)$ to be the equivalence set of non–abelian homomorphisms $\pi_1(X(K))\to S_k$ up to conjugation by elements in S_k. Let $\alpha_1,\alpha_2:\pi_1(X(K))\to S_k$ be representatives of the same element in $R_k(K)$. Clearly

$$H_1^{\overline{\alpha_1}}(X(K),R^k[t^{\pm1}])\cong H_1^{\overline{\alpha_2}}(X(K),R^k[t^{\pm1}])$$

as $R[t^{\pm1}]$–modules. In particular

$$\Delta_K^k(t):=\prod_{[\alpha]\in R_k}\Delta_K^{\overline{\alpha}}(t)\in R[t^{\pm1}]$$

is an invariant of the knot K, well–defined up to a unit in $R[t^{\pm1}]$. If there are no non–abelian homomorphisms $\pi_1(X(K))\to S_k$, i.e. when $R_k(K)=\emptyset$, then we set $\Delta_K^k(t)=1$.

Remark.
(1) Clearly the set of twisted Alexander polynomials corresponding to all $[\alpha]\in R_k$ is a knot invariant as well (up to the appropriate indeterminacy). For purely notational purposes we prefer the above invariant, even though it is slightly weaker.
(2) This definition can be modified in many different ways, we picked this definition, since it is simple enough to be easily implemented by a computer program, but strong enough to distinguish many interesting knots.
(3) This definition can immediately generalized to give an invariant of a pair (N,ϕ) where N is a compact manifold and $\phi:\pi_1(N)\to F$ is a homomorphism to a finitely generated free abelian group.
(4) It is known that for $n\neq 6$ all automorphisms of S_n are inner automorphisms, i.e. given by conjugation by an element in S_n. In particular considering equivalence classes of representations up to automorphisms of S_k gives the same polynomials for $k\neq 6$.

4. Computing twisted Alexander polynomials

In this section we will show how Fox calculus can be used to determine $\Delta_1^\alpha(t)$ and $\tilde{\Delta}_1^\alpha(t)$ efficiently. We refer to [Fo53], [Fo54] and [CF77] for more information on Fox calculus. Let M be a 3–manifold and let $\langle g_1,\dots,g_s|r_1,\dots,r_q\rangle$ be a presentation of $\pi_1(M)$. Let $\phi\in H^1(M;\mathbb{Z})$ and $\alpha:\pi_1(M)\to\mathrm{GL}(\mathbb{F},k)$ a representation. In this section we will show how to compute $\Delta_1^\alpha(t)$ and $\Delta_0^\alpha(t)$.

First, $\Delta_1^\alpha(t)$ can be computed using Fox calculus as follows. By [CF77, p. 98] there exist unique maps $\partial_i : \langle g_1,\dots,g_s\rangle \to \mathbb{Z}\langle g_1,\dots,g_s\rangle$ such that

$$\begin{array}{rcll} \partial_i(g_j) &=& \delta_{ij}, & \text{for any } i,j, \\ \partial_i(uv) &=& \partial_i(u)+u\partial_i(v), & \text{for any } u,v\in\langle g_1,\dots,g_s\rangle. \end{array}$$

This gives indeed a well–defined map. Denote by $\bar{f}$ for $f\in\mathbb{Z}[\pi_1(M)]$ the involution induced by $\bar{g}=g^{-1}$ for any $g\in\pi_1(M)$. Then apply the map

$$\phi\otimes\alpha : \mathbb{Z}[\pi_1(M)] \to M_{k\times k}(\mathbb{F}[t^{\pm1}])$$

to the entries of the $s\times q$–matrix $(\overline{\partial_i(r_j)})$. We denote the resulting $sk\times qk$–matrix over $\mathbb{F}[t^{\pm1}]$ by A. Since $\mathbb{F}[t^{\pm1}]$ is a PID we can do row and column operations to get A into the following form

$$\begin{pmatrix} p_1(t) & 0 & \dots & 0 & 0 \\ 0 & p_2(t) & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & p_l(t) & 0 \\ 0 & 0 & \dots & 0 & (0)_{ks-l\times kq-l} \end{pmatrix}$$

where $p_i(t)\in\mathbb{F}[t^{\pm1}]\setminus\{0\}$.

Theorem 4.1. *If $l<k(s-1)$ then $\Delta_1^\alpha(t)=0$. Otherwise*

$$\Delta_1^\alpha(t)=\prod_{i=1}^{l} p_i(t).$$

Furthermore $\tilde{\Delta}_1^\alpha(t)=\prod_{i=1}^{l} p_i(t)$.

Proof. Write $\pi:=\pi_1(M)$ and $K:=K(\pi,1)$. Note that $H_1^\alpha(M,\mathbb{F}^k[t^{\pm1}])\cong H_1^\alpha(K,\mathbb{F}^k[t^{\pm1}])$ (cf. [FK05]). It therefore suffices to compute the latter homology.

Note that we can assume that K has one 0–cell, s 1–cells corresponding to the generators $g_1,\dots,g_s$ and q 2–cells corresponding to the relations $r_1,\dots,r_q$. Denote the universal cover of K by $\tilde{K}$. Let $p\in K$ be the point corresponding to the 0–cell. Denote the preimage of p under the map $\tilde{K}\to K$ by $\tilde{p}$. Note that $C_i(\tilde{K},\tilde{p})=C_i(\tilde{K})$ for $i\geq 0$. We therefore get an exact sequence

$$C_2(\tilde{K})\otimes_{\mathbb{Z}[\pi]}\mathbb{F}^k[t^{\pm1}] \xrightarrow{d_2\otimes id} C_1(\tilde{K})\otimes_{\mathbb{Z}[\pi]}\mathbb{F}^k[t^{\pm1}] \to H_1^\alpha(K,p;\mathbb{F}^k[t^{\pm1}])\to 0.$$

The equivariant lifts of the cells give $\mathbb{Z}[\pi]$–bases for $C_2(\tilde{K})$ and $C_1(\tilde{K})$. As Harvey [Ha05, Section 6] pointed out, the $\mathbb{Z}[\pi]$–right module homomorphism $d_2 : C_2(\tilde{K})\to C_1(\tilde{K})$ with respect to these bases is given by the $s\times q$–matrix $(\overline{\partial_i(r_j)})$. Clearly $A=(\phi\otimes\alpha)(\overline{\partial_i(r_j)})$ now represents $d_2\otimes_{\mathbb{Z}[\pi]}\text{id}$. Therefore A is a presentation matrix for $H_1^\alpha(K,p;\mathbb{F}^k[t^{\pm1}])$.

Now consider the following diagram whose rows are exact:

$$\begin{array}{ccccccc} 0\to H_1^\alpha(K,\mathbb{F}^k[t^{\pm 1}])\to & H_1^\alpha(K,p,\mathbb{F}^k[t^{\pm 1}]) & \to H_0^\alpha(p,\mathbb{F}^k[t^{\pm 1}])\to H_0^\alpha(K,\mathbb{F}^k[t^{\pm 1}]) \\ \| & \downarrow\cong & \downarrow\cong \quad\quad \| \\ 0\to H_1^\alpha(K,\mathbb{F}^k[t^{\pm 1}])\to \bigoplus_{i=1}^{l}\mathbb{F}[t^{\pm 1}]/(p_i(t))\oplus\mathbb{F}^{ks-l}[t^{\pm 1}]\to & \mathbb{F}^k[t^{\pm 1}] & \to H_0^\alpha(K,\mathbb{F}^k[t^{\pm 1}]). \end{array}$$

Consider $H_0^\alpha(K,\mathbb{F}^k[t^{\pm 1}]) = \mathbb{F}^k[t^{\pm 1}]/\{g\cdot v - v | g\in\pi, v\in\mathbb{F}^k[t^{\pm 1}]\}$. The action of π is given by $\phi\otimes\alpha$. Since ϕ is non–trivial it follows that $H_0^\alpha(K,\mathbb{F}^k[t^{\pm 1}])$ is a finite–dimensional vector space over $\mathbb{F}$. It follows that the kernel of the homomorphism $\mathbb{F}^k[t^{\pm 1}]\to H_0^\alpha(K;\mathbb{F}^k[t^{\pm 1}])$ is isomorphic to $\mathbb{F}^k[t^{\pm 1}]$ again. Putting all these together it follows that if $ks-l>k$ then $\Delta_1^\alpha(t)=0$, otherwise $\Delta_1^\alpha(t)=\prod_{i=1}^l p_i(t)$. Clearly it also follows that $\tilde{\Delta}_1^\alpha(t)=\prod_{i=1}^l p_i(t)$. □

Now apply $\phi\otimes\alpha$ to the $1\times s$–matrix $(1-g_1^{-1},\dots,1-g_s^{-1})$. Denote the resulting $k\times sk$–matrix by B. Since $\mathbb{F}[t^{\pm 1}]$ is a PID we can do row and column operations to get B into the following form

$$\begin{pmatrix} q_1(t) & \dots & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 & \dots & 0 \\ 0 & \dots & q_k(t) & 0 & \dots & 0 \end{pmatrix}$$

where $q_i(t)\in\mathbb{F}[t^{\pm 1}]$.

Lemma 4.2.

$$\Delta_0^\alpha(t)=\prod_{i=1}^k q_i(t).$$

Proof. We use the same notation as in the proof of the previous lemma. We have an exact sequence

$$C_1(\tilde{K})\otimes_{\mathbb{Z}[\pi]}\mathbb{F}^k[t^{\pm 1}]\xrightarrow{d_1\otimes id} C_0(\tilde{K})\otimes_{\mathbb{Z}[\pi]}\mathbb{F}^k[t^{\pm 1}]\to H_0^\alpha(K;\mathbb{F}^k[t^{\pm 1}])\to 0.$$

Consider the $\mathbb{Z}[\pi]$–right module homomorphism $d_1: C_1(\tilde{K})\to C_0(\tilde{K})$ together with the bases given by cells. Then ∂_1 is represented by the $1\times s$–matrix $(1-g_1^{-1},\dots,1-g_s^{-1})$. Therefore B is a presentation matrix for $H_0^\alpha(K;\mathbb{F}^k[t^{\pm 1}])$. □

Remark. All the computations can be done over the ring $\mathbb{F}[t^{\pm 1}]$. Therefore we can apply the Euclidean algorithm to quickly find a 'diagonal' form for the matrix A.

5. Examples

5.1. **Representations of 3–manifold groups.** Given a presentation $\langle g_1,\dots,g_s|r_1,\dots,r_t\rangle$, of $\pi_1(M)$ for a 3-manifold M finding a representation to $\mathrm{GL}(\mathbb{F},k)$ for some k is easy in theory: it is enough to assign arbitrary elements in $\mathrm{GL}(\mathbb{F},k)$ to $g_1,\dots,g_s$ and check whether these satisfy the relations. Our experience shows that this is not an effective way of finding representations since $\mathrm{GL}(\mathbb{F}, k)$ has approximately p^{k^2} elements, and therefore there are $s^{p^{k^2}}$ possible assignments of elements in $\mathrm{GL}(\mathbb{F}, k)$ to s generators.

In our applications we therefore first find homomorphisms $\pi_1(M) \to G$, G a finite group and then find a representation of $\mathbb{F}[G]$. In most cases we take $G = S_k$ for some k, but metabelian groups can also be useful.

Clearly S_k acts on $\mathbb{F}^k$ by permutation. But S_k also has another very interesting representation. Indeed, if $\varphi : \pi_1(M) \to S_k$ is a homomorphism then we can consider

$$\alpha(\varphi) : \pi_1(M) \xrightarrow{\varphi} S_k \to \mathrm{GL}(V_{k-1}),$$

where

$$V_k := \{(v_1, \dots, v_{k+1}) \in \mathbb{F}_{13}^{k+1} | \sum_{i=1}^{k+1} v_i = 0\}.$$

Clearly $\dim(V_k) = k$ and S_{k+1} acts on it by permutation. These representations are easy to find and remarkably useful for our purposes. Note that V_k is a subrepresentation of a unitary representation, hence $\alpha(\varphi)$ is unitary as well.

5.2. **Knots with up to 12 crossings: genus bounds and fiberedness. I. Knot genera :** It turns out that for all knots with 10 crossings or less we have

$$2\,\mathrm{genus}(K) = \deg(\Delta_K(t)).$$

On the other hand it is known that

$$2\,\mathrm{genus}(K) > \deg(\Delta_K(t))$$

for many knots with more than 10 crossings. Perhaps the most famous for which this inequality is strict is $K = 11^n_{34}$ (the Conway knot). This knot has Alexander polynomial one, i.e. the degree of $\Delta_K(t)$ equals zero. Furthermore this implies that $\pi_1(X(K))^{(1)}$ is perfect, i.e. $\pi_1(X(K))^{(n)} = \pi_1(X(K))^{(1)}$ for any $n > 1$. Therefore the genus bounds of Cochran [Co04] and Harvey [Ha05] vanish as well.

FIGURE 1. The Conway knot 11^n_{34} and a Seifert surface of genus 3 (from [Ga84]).

The fundamental group of $\pi_1(X(K)) = \pi_1(S^3 \setminus K)$ is generated by the meridians $a, b, \dots, k$ of the segments in the knot diagram of Figure 1. The relations are

$$\begin{array}{rclrclrclrcl} a &=& jbj^{-1}, & b &=& fcf^{-1}, & c &=& g^{-1}dg, & d &=& k^{-1}ek, \\ e &=& h^{-1}fh, & f &=& igi^{-1}, & g &=& e^{-1}he, & h &=& c^{-1}ic, \\ i &=& aja^{-1}, & j &=& iki^{-1}, & k &=& e^{-1}ae. & & & \end{array}$$

Using the program *KnotTwister* we found the homomorphism $\varphi : \pi_1(X(K)) \to S_5$ given by

$$\begin{array}{rclrclrclrcl} A &=& (142), & B &=& (451), & C &=& (451), & D &=& (453), \\ E &=& (453), & F &=& (351), & G &=& (351), & H &=& (431), \\ I &=& (351), & J &=& (352), & K &=& (321), & & & \end{array}$$

where we use cycle notation generators. The generators of $\pi_1(X(K))$ are sent to the element in S_5 given by the cycle with the corresponding capital letter. We then consider $\alpha := \alpha(\varphi) : \pi_1(X(K)) \xrightarrow{\varphi} S_5 \to \mathrm{GL}(V_4(\mathbb{F}_{13}))$. Using *KnotTwister* we compute $\deg(\Delta_0^\alpha(t)) = 0$ and we compute the Alexander polynomial to be

$$\Delta_1^\alpha(t) = 1 + 6t + 9t^2 + 12t^3 + t^5 + 3t^6 + t^7 + 3t^8 + t^9 + 12t^{11} + 9t^{12} + 6t^{13} + t^{14} \in F_{13}[t^{\pm 1}].$$

Note that Theorem 2.1 says that if $\Delta_1^\alpha(t) \neq 0$, then

$$\mathrm{genus}(K) \geq \frac{1}{2}\left(\frac{1}{k}\deg(\Delta_1^\alpha(t)) - \frac{1}{k}\deg(\Delta_0^\alpha(t)) + 1\right).$$

In our case we therefore get

$$\mathrm{genus}(K) \geq \frac{1}{2}\left(\frac{14}{4} + 1\right) = \frac{18}{8} = 2.25.$$

Since $\mathrm{genus}(K)$ is an integer we get $\mathrm{genus}(K) \geq 3$. Since there exists a Seifert surface of genus 3 for K (cf. [Ga84] and Figure 1) it follows that the genus of the Conway knot is three. All the other knots with 12 crossings or less for which $\mathrm{genus}(K) > \frac{1}{2}\deg \Delta_K(t)$ are among the list of examples provided by *KnotTwister*.

II. Fiberedness: It is known that a knot with 11 or fewer crossings is fibered if and only if the Alexander polynomial is monic and $\deg(\Delta_1(t)) = 2\,\mathrm{genus}(K)$. According to Stoimenow [Sto] there are 52 12–crossing knots which have monic Alexander polynomials and such that $\deg(\Delta_1(t)) = 2\,\mathrm{genus}(K)$. Hirasawa showed that among these the knots 12_{1498}, 12_{1502}, 12_{1546} and 12_{1752} are not fibered.

Consider the knot $K = 12_{1345}$. Its Alexander polynomial equals $\Delta_1(t) = 1 - 2t + 3t^2 - 2t^3 + t^4$ and its genus equals two. K therefore has the abelian invariants of a fibered knot. It follows from Corollary 2.7 that if K was fibered, then for any field $\mathbb{F}$ and any representation $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ the following holds:

$$\deg(\Delta_1(t)) = \frac{1}{k}\deg(\Delta_1^\alpha(t)) + \left(1 - \frac{1}{k}\deg(\Delta_0^\alpha(t))\right).$$

We found a representation $\alpha : \pi_1(X(K)) \to S_4 \to \mathrm{GL}(\mathbb{F}_3, 4)$ such that $\deg(\Delta_1^\alpha(t)) = 7$ and $\deg(\Delta_0^\alpha(t)) = 1$. We compute

$$\frac{1}{4}\deg(\Delta_1^\alpha(t)) + \left(1 - \frac{1}{4}\deg(\Delta_0^\alpha(t))\right) = \frac{10}{4} \neq 4.$$

Hence K is not fibered.

Altogether 13 knots are not fibered, they are among the list of examples provided by *KnotTwister*. Stoimenow and Hirasawa then showed that the remaining 39 knots are in fact fibered. So Corollary 2.7 was crucial in finding all non–fibered 12–crossing knots.

III. Mutants: The program *KnotTwister* comes with the list of all mutants pairs and triples with 12 crossings or less. The knots in these groups can all be distinguished using $\Delta_K^5(t)$.

6. Using *KnotTwister*

The user has to choose a knot or the fundamental group π of a 3–manifold M together with a homomorphism $\phi : \pi \to \mathbb{Z}$. Furthermore the user has to choose the type of representation *KnotTwister* should consider. *KnotTwister* will then attempt to find representations of the given type. Once it finds a representation *KnotTwister* will compute the Alexander polynomials and find the corresponding bounds on the knot genus, respectively on the Thurston norm of a given $\phi \in H^1(M;\mathbb{Z})$.

6.1. **Entering braids and groups.** *KnotTwister* can work with three types of describing knots and manifolds: braid descriptions and two types of presentations for the fundamental group.

Braid description: A braid description is always of the form $\{a_1, \dots, a_k\}$ where $a = +i$ stands for a positive crossing of the strands strands i and $i+1$ and $a = -i$ stands for a negative crossing of the strands i and $i+1$. For example $\{1, 1, 1\}$ stands for a braid with two strands and three positive crossings, i.e. it describes the left-handed trefoil, whereas $\{-1, -1, -1\}$ stands for the right handed trefoil. The figure 8 knot is given by $\{-1, 2, -1, 2\}$.

The braid descriptions for all knots with up to 11 crossings are from Alexander Stoimenow's webpage [Sto] which also provides the braid descriptions for all 12 crossing knots.

Group presentation (I): A second possibility is to write a group presentation into a file and then load it while running *KnotTwister*. The convention is that generators have to be lower case letters from a to z or upper case letters from A to Z (the program is case sensitive), the relators have to be written 'additively'. It is perhaps easiest to consider an example:

If the file contains the two lines

$$-a-c+b+c$$
$$-b-a+c+a$$

then *KnotTwister* will read this as the presentation

$$\langle a,b,c|a^{-1}c^{-1}bc, b^{-1}a^{-1}ca\rangle.$$

This is a Wirtinger presentation for the fundamental group of the trefoil knot. In particular the generators do not have to be specified, *KnotTwister* assumes that the letters appearing in the relations are all the generators.

In order to define Alexander polynomials a homomorphism $\phi : \pi_1(M) \to \mathbb{Z}$ has to be specified. In the case that only the relators are given in the file, then *KnotTwister* will assume that ϕ sends all generators to $1 \in \mathbb{Z}$.

The homomorphism ϕ can also be specified explicitly. For example if the file consists of the lines

$$-a-b+a+b$$
$$-b-a+b+a$$
$$a0$$
$$b-1$$

then *KnotTwister* will read this as the presentation

$$\langle a,b|a^{-1}b^{-1}ab, b^{-1}a^{-1}ba\rangle$$

together with the homomorphism ϕ defined by $\phi(a)=0$, $\phi(b)=-1$.

Group presentation (II): A second to write a group presentation is given by multiplicative notation, the presentation has to be written as in the following example:

$$x\_\{3\}x\_\{2\}x\_\{3\}\hat{}\{-1\}x\_\{1\}\hat{}\{-1\}$$
$$x\_\{1\}x\_\{3\}x\_\{1\}\hat{}\{-1\}x\_\{2\}\hat{}\{-1\}$$
$$x\_\{1\}x\_\{4\}x\_\{1\}\hat{}\{-1\}x\_\{3\}\hat{}\{-1\}$$
$$x\_\{3\}x\_\{1\}x\_\{3\}\hat{}\{-1\}x\_\{4\}\hat{}\{-1\}.$$

Such a group presentation is for example generated by Kodama's computer program 'KnotGTK'. This program can be downloaded from

`http://www.math.kobe-u.ac.jp/~kodama/knot.html`

Using 'KnotGTK' it is possible to draw a knot or link, then compute a presentation of the knot group which will be of the above type, save it, and then load it into *KnotTwister* . For Windows users, the way to save the data from 'KnotGTK' is to click on the right mouse button on the top frame, then select all and copy into a text file.

KnotTwister tries to give the correct Thurston norm bounds using the appropriate theorem. The user therefore has to specify which of the following best describes the situation in the file:

(1) the group is the fundamental group of a knot complement,
(2) the group is the fundamental group of a link complement,
(3) the group is the fundamental group of a 3-manifold with boundary,
(4) the group is the fundamental group of a 3-manifold without boundary.

6.2. **Different types of representations.** The user picks a finite field $\mathbb{F}$, the preset choice is $\mathbb{F} = \mathbb{F}_{13}$. *KnotTwister* can then find representations of the following type:

(1) Representations of the form
$$\pi_1(M) \to S_k \to GL(\mathbb{F}, k),$$
where the symmetric group S_k acts on $\mathbb{F}^k$ by permutation. The size k can be chosen by the user. If *KnotTwister* does not find representations for a given k, *KnotTwister* will increase k. Clearly such a representation is unitary.

(2) Representations of the form
$$\pi_1(M) \to S_k \to GL(V_{k-1}),$$
where $V_l := \{(v_1, ..., v_{l+1}) \in \mathbb{F}^{l+1} | \sum_{i=1}^{l+1} = 0\}$. Note that V_l is vector space of dimension l and the symmetric group S_k acts on V_{k-1} by permutation. Note that $V_{k-1} \subset \mathbb{F}^k$ on which S_k acts unitarily. Hence S_k also acts unitarily on V_{k-1}.

(3) Abelian representations of the form
$$\pi_1(M) \to GL(\mathbb{F}, 1).$$
These representations do not give any interesting information for knots, but can sometimes be useful for studying the Thurston norm of links (cf. [FK05] for an example).

(4) Representations of the form
$$\pi_1(M) \to S_k.$$
In this case the *KnotTwister* computes the image G of $\pi_1(M)$ in S_k and then considers the representation
$$\pi_1(M) \to G \to \mathrm{Aut}(\mathbb{F}[G])$$
where G acts on the group ring of G by left-multiplication. Note that these tend to be representations of high dimension since $\dim_{\mathbb{F}}(\mathbb{F}[S_k]) = k!$. These representations can give obstructions to the manifold $S^1 \times M$ being symplectic (cf. [FV06a]). Clearly such a representation is unitary.

(5) Metabelian representations. Let l be a prime number and let $z \in \mathbb{Z}/l$ be such that $z^n = 1$, then we can study representations of the form
$$\alpha : \pi_1(M) \to \mathbb{Z}/n \ltimes \mathbb{Z}/l.$$
Here $1 \in \mathbb{Z}/n$ acts on b in $\mathbb{Z}/l$ by multiplication by z. Clearly such a representation is unitary.

(6) Representations factoring through metabelian representations. Let l be a prime number and let $z \in \mathbb{Z}/l$ be such that $z^n = 1$, then we can study representations of the form

$$\alpha : \pi_1(M) \to \mathbb{Z}/n \ltimes \mathbb{Z}/l \to \mathrm{GL}(\mathbb{F}[\mathbb{Z}/l]).$$

Here $1 \in \mathbb{Z}/l$ acts on $b \in \mathbb{Z}/l$ by addition of one, and $1 \in \mathbb{Z}/n$ acts on $b \in \mathbb{Z}/l$ by multiplication by z. Clearly such a representation is unitary.

For finding lower bounds on the Thurston norm the representations of the form $\pi_1(M) \to S_k \to GL(V_{k-1})$ tend to be most useful. They are easy to find, the twisted Alexander polynomials are easily computed and the term $\deg(\Delta_0^\alpha(t))$ tends to be small.

If the goal is to check whether a knot is fibered, or whether a manifold fibers, then the best approach tends to be to consider representations of the form $\pi_1(M) \to S_k \to \mathrm{GL}(\mathbb{F}, k)$ and to allow the *KnotTwister* to use different fields. In that case *KnotTwister* will find a homomorphism $\pi_1(M) \to S_k$ and compute the Alexander polynomials corresponding to $\pi_1(M) \to S_k \to \mathrm{GL}(\mathbb{F}, k)$ for different fields.

Given a pair (M, ϕ) and a representation type, *KnotTwister* will first try to find a presentation will a small number of generators, but in most cases it does not find the optimal representation. Then *KnotTwister* will randomly assign elements in the chosen group to generators of $\pi_1(M)$ and check whether the relations are satisfied. Note that the size of the group S_k is $k!$, in particular for $k > 6$ *KnotTwister* takes much more time to find representations. The minimal number of generators of course also plays a big role.

6.3. **Lists of knots.** Often it is convenient to deal with a large number of knots at once. *KnotTwister* can therefore also deal with lists of braid descriptions. A typical file has to be of the following form:

```
[ name of knot 1]    (braid description)
[ name of knot 2]    (braid description)
end
```

If a line starts with '#' then this line will be ignored by KnotTwister, furthermore if a line contains 'new' KnotTwister will redo the computations.

KnotTwister will try to find polynomials for all the knots, it will record the Alexander polynomial, the highest genus bound and the fiberedness information KnotTwister could find. In case of representations factoring through S_k, the program will try to find representations for each knot in the list, before starting over with increased k. Working on a list can be interrupted at any moment, the program will continue where it stopped.

For example consider the following list:

3_1	$-1,-1,-1$
4_1	$-1,2,-1,2$
11_{401}	$1,2,2,-3,1,-3,1,-2,1,-2,-3$
end	

As an example we could get the following output:

3_1 :	$1+12t+t^2$	reps of dim 3y4y5y	genus ≥ 1.00	perhaps fibered	generators ≥ 2
4_1 :	$1+10t+t^2$	reps of dim 3n4y5n	genus ≥ 1.00	perhaps fibered	generators ≥ 2
5_1 :	1	reps of dim 3n4n5y	genus ≥ 2.25	not fibered	generators ≥ 2

This shows that KnotTwister computed the untwisted Alexander polynomial (over the chosen finite field), it shows that it tried to find representations factoring through S_k for $k = 3, 4, 5$ for all four knots. Note that 3y means it found a representation factoring through S_3, whereas 3n means it did not find a representation factoring through S_3. The computation KnotTwister will do next is to try to find representations factoring through S_6 for the first knot.

The genus bounds found so far are given. KnotTwister found that the last knot is not fibered, note that KnotTwister can only make negative statements about fiberedness, it can not show that a given knot is fibered. Generators ≥ 2 means that the minimal number of generators for $\pi_1(S^3 \setminus K)$ is at least 2.

Note: If more than one field is chosen, then KnotTwister assumes that the user is only interested in fiberedness, this means, that if KnotTwister establishes that a knot is not fibered it will move on to the next knot, and will not attempt to find better genus bounds.

6.4. **Options.** Given a braid the user can either work with the knot K or link L which is given by closing the braid, or with the manifold given by 0–framed surgery along K or the first component of L.

KnotTwister can also find the size of the image of $\pi_1(M)$ in S_k, furthermore it can find the minimal number of generators of the image. This number gives a lower bound on the minimal number of generators of $\pi_1(M)$.

All the results can be saved to the file '`KnotTwister_results.txt`'. Furthermore the matrix $(\phi\otimes\alpha)(\overline{\partial_i(r_j)})$ can be saved to the file '`KnotTwister_matrix.txt`'. Finally the matrix $(\phi\otimes\alpha)(\overline{\partial_i(r_j)})$ can be saved in such a form that GAP [GAP] can compute its diagonal form, the filename is '`KnotTwister_matrix_gap.txt`'.

References

[Ch03] J. Cha, *Fibred knots and twisted Alexander invariants*, Transactions of the AMS 355: 4187–4200 (2003)

[Co04] T. Cochran, *Noncommutative knot theory*, Algebraic and Geometric Topology 4, paper no. 19: 347-398 (2004)

[CF77] R. Crowell, R. Fox, *Introduction to knot theory*, Graduate Text in Mathematics 57, Springer Verlag (1977)

[Fo53] R. H. Fox, *Free differential calculus I, derivation in the free group ring*, Ann. of Math. Soc. 57:547–560 (1953)

[Fo54] R. H. Fox, *Free differential calculus II, the isomorphism problem*, Ann. of Math. Soc. 59:196–210 (1953)

[FK05] S. Friedl, T. Kim, *The Thurston norm and twisted Alexander polynomials*, Preprint, to appear in Topology (2005)

[FV06a] S. Friedl, S. Vidussi, *Twisted Alexander polynomials and symplectic structures*, preprint (2006)

[FV06b] S. Friedl, S. Vidussi, *Non–trivial Alexander polynomials of knots and links*, preprint (2006)

[Ga84] D. Gabai, *Foliations and genera of links*, Topology 23, No. 4: 381–394 (1984)

[GAP] *GAP*, `http://www-gap.mcs.st-and.ac.uk/`

[Ha05] S. Harvey, *Higher–order polynomial invariants of 3–manifolds giving lower bounds for the Thurston norm*, Topology **44** (2005), 895–945.

[Hi02] J. Hillman, *Algebraic invariants of links*, Series on Knots and Everything, 32. World Scientific Publishing Co., Inc., River Edge, NJ (2002)

[JW93] B. Jiang, S. Wang, *Twisted topological invariants associated with representations*, Topics in knot theory (Erzurum 1992), 211–227, NATO Adv. Sci. Inst. C. Math. Phys. Sci, 399, Kluwer Acad. Publ., Dordrecht (1993)

[KL99a] P. Kirk, C. Livingston, *Twisted Alexander invariants, Reidemeister torsion and Casson–Gordon invariants*, Topology 38, no. 3: 635–661 (1999)

[KL99b] P. Kirk, C. Livingston, *Twisted knot polynomials: inversion, mutation and concordan*, Topology 38, no. 3: 663–671 (1999)

[Kit96] T. Kitano, *Twisted Alexander polynomials and Reidemeister torsion*, Pacific J. Math. 174: 431–442 (1996)

[Lic97] W. B. R. Lickorish, *An introduction to knot theory*, Springer Graduate Texts in Mathematics 175 (1997)

[Lin01] X. S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin. (Engl. Ser.) 17, no. 3: 361–380 (2001)

[Mc02] C. T. McMullen, *The Alexander polynomial of a 3–manifold and the Thurston norm on cohomology*, Ann. Sci. Ecole Norm. Sup. (4) 35, no. 2: 153–171 (2002)

[OS04a] P. Ozsváth, Z. Szabó, *Holomorphic disks and genus bounds*, Geometry & Topology, vol 8: 311–334 (2004)

[OS04b] P. Ozsváth, Z. Szabó, *Knot Floer homology, genus bounds, and mutation*, Topology Appl. 141, no. 1-3: 59–85 (2004)

[Sto] A. Stoimenow, `http://www.ms.u-tokyo.ac.jp/~ stoimeno/ptab/index.html`

[Th86] W. P. Thurston, *A norm for the homology of 3–manifolds*, Mem. Amer. Math. Soc., 339: 99–130 (1986)

[Tu02b] V. Turaev, *Torsions of 3–manifolds*, Progress in Mathematics, 208, Birkhäuser, Basel (2002)

[Vi99] S. Vidussi, *The Alexander norm is smaller than the Thurson norm; a Seiberg–Witten proof.*, Prepublication Ecole Polytechnique 6 (1999)

[Vi03] S. Vidussi, *Norms on the cohomology of a 3-manifold and SW theory*, Pacific J. Math. 208, no. 1: 169–186 (2003)

[Wa94] M. Wada, *Twisted Alexander polynomial for finitely presentable groups*, Topology 33, no. 2: 241–256 (1994)

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