## Integration theorems

If $C$ is an oriented curve and $f(x, y, z)$ is a function, then

$$\int_C \nabla f \cdot d\mathbf{S} = f(\text{final point of } C) - f(\text{starting point of } C).$$

Note that the two end points of a curve can be viewed as the boundary of the curve. This result then already encapsulates the principle of the more complicated integration theorems: If $X$ is a curve, surface, solid, then

$$\int_X \text{derivative of } G = \int_{\partial(X)} G$$

the trick is to know what $G$ can be (function or vector field?) and what the 'derivative' is in each situation.

**Green's theorem:** Let $S$ be a region in $\mathbb{R}^2$ and $\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y))$ a vector field. Then

$$\int\int_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \int_{\partial(S)} \mathbf{F}(x, y)\, d\mathbf{S}.$$

The boundary $\partial(S)$ has the following orientation: Walking along the orientation the surface has to be on the left.

Green's theorem can be used as follows:

1. An integral $\int\int_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$ can be replaced by a line integral. The most common application is $\mathbf{F} = \frac{1}{2}(-y, x)$. In that case

$$\int\int_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \int\int_S 1\, dA = \text{area of } S.$$

2. A more important application is to go the other way: replace a line integral by an area integral. This works if the line integral is over a *closed* curve. Green's theorem comes in handy if closed curve is difficult to parametrize, e.g. the boundary of a rectangle, whereas the area it bounds is easy to parametrize.

**Stokes' theorem:** Let $S$ be an oriented surface (i.e. you pick a normal vector, indicating 'outside') and $\mathbf{F}(x, y, z)$ a vector field. Then

$$\int\int_S \nabla \times \mathbf{F}\, d\mathbf{S} = \int_{\partial(S)} \mathbf{F}(x, y, z)\, d\mathbf{S}.$$

The boundary $\partial(S)$ has the following orientation: Take your thumb of your right hand and let it point into the normal direction of the oriented surface, then your fingers will give the direction of the boundary.

1. Stokes' theorem is mostly used to replace the integral of a vector field of the form $\nabla \times \mathbf{F}$ by a line integral. Line integrals tend to be much easier than integrals of vector fields over surfaces.

2. Stokes' theorem also makes it possible to replace the integral of a vector field of the form $\nabla \times \mathbf{F}$ over a difficult surface by the integral of $\nabla \times \mathbf{F}$ over a nice surface with the same boundary.

3. Note that if $S$ has no boundary, e.g. if $S$ is a sphere or the boundary of a cube, then $\int\int \nabla \times \mathbf{F}\, d\mathbf{S} = 0$.

**Gauss theorem:** Let $R$ be a solid and $\mathbf{F}(x, y, z)$ a vector field. Then

$$\int\int\int_R \mathrm{div}(\mathbf{F}(x, y, z))\, dV = \int\int_{\partial(R)} \mathbf{F}(x, y, z)\, d\mathbf{S}.$$

Here we give $\partial(R)$ the normal vector which points away from $R$.

Gauss theorem is mostly used to convert a flux integral $\int\int_S \mathbf{F}(x, y, z)\, d\mathbf{S}$ over a *closed* surface which might be difficult to parametrize (e.g. the boundary of a cube) into a triple integral over a solid which bounds the surface (e.g. the cube) and which often is much easier to parametrize.

Summarizing we get the following integration theorems in $\mathbb{R}^3$:

$$\begin{aligned} \int\int\int_R \operatorname{div}(\mathbf{F})\, dV &= \int\int_{\partial(R)} \mathbf{F}\, d\mathbf{S} && \text{Gauss' theorem} \\ \int\int_S \nabla \times \mathbf{F}\, d\mathbf{S} &= \int_{\partial(S)} \mathbf{F}\, d\mathbf{S} && \text{Stokes' theorem} \\ \int_C \nabla f \cdot d\mathbf{S} &= f(\text{end point of } C) - f(\text{starting point of } C). \end{aligned}$$

One can easily show that

$$\begin{aligned} \nabla \times (\nabla f) &= 0 \\ \operatorname{div}(\nabla \times \mathbf{F}) &= 0 \end{aligned}$$

i.e. the application of two consecutive 'derivatives' give zero. Here 'consecutive' means: derivatives which appear in consecutive integration theorems. For example it is not true that $\operatorname{div}\nabla f = 0$ for any function $f$.

This observation that the application of two consecutive 'derivatives' gives zero is in fact related to the fact that if one takes twice the boundary of any geometric object, then one gets nothing. For example the boundary of a ball is the sphere, but the boundary of a sphere is empty.