## Homework 13 – Solutions

2. Let $\mathbf{F}(x,y,z) = (x^2, y-z, 1)$. Let $S$ be the graph of $f(x,y) = x^3 - 2xy$ with domain $D = [0,1] \times [0,3]$. Compute $\int\int_S \mathbf{F} \cdot d\mathbf{S}$.
Solution: We first need a parametrization of $S$. We can describe $S$, since it's a graph, in terms of $x, y$. We get
$$\begin{aligned} x &= x \\ y &= y \\ z &= x^3 - 2xy. \end{aligned}$$
with $x \in [0,1]$ and $y \in [0,3]$ (from the domain of $f$). So we get
$$\Phi(x,y) = (x, y, x^3 - 2xy).$$
We compute
$$\begin{aligned} T_x &= (1, 0, 3x^2 - 2y) \\ T_y &= (0, 1, -2x) \\ T_x \times T_y &= (-3x^2 + 2y, 2x, 1). \end{aligned}$$
So we get
$$\begin{aligned} \int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int_{x=0}^{x=1} \int_{y=0}^{y=3} \mathbf{F}(\Phi(x,y)) \cdot (T_x \times T_y) dxdy \\ &= \int_{x=0}^{x=1} \int_{y=0}^{y=3} \mathbf{F}(x, y, x^3 - 2xy) \cdot (-3x^2 + 2y, 2x, 1) dxdy \\ &= \int_{x=0}^{x=1} \int_{y=0}^{y=3} (x^2, y - x^3 + 2xy, 1) \cdot (-3x^2 + 2y, 2x, 1) dxdy \\ &= \int_{x=0}^{x=1} \int_{y=0}^{y=3} -3x^4 + 2x^2y + 2xy = 2x^4 + 4x^2y + 1 dxdy. \end{aligned}$$

3. Let $C$ be the coke can with radius 3 and with top and bottom at $z = 1$ and $z = 0$. Give it the usual orientation. Let $F(x,y,z) = (0,0,z)$. Compute $\int\int_C \mathbf{F} d\mathbf{S}$. If you think about what is happening geometrically, then you should be able to solve this problem with barely any computations. Explain carefully what you are doing. (Hint: sketch $C$ and $\mathbf{F}$).
Solution: $\mathbf{F}$ is parallel to the side of the cylinder, so nothing flows in nor out. Now note that at the bottom $z = 0$, so the vector field is zero, and the flow is hence zero. At the top we have $F(x,y,z) = (0,0,1)$ since $z = 1$. Since the flow is perpendicular to the top we see that the flux is just 1 times the area of the top, so we get $1 \cdot 3^2\pi = 9\pi$.