Homework 14 – Solutions

1. Compute $\int_C \mathbf{F}(x,y)\cdot d\mathbf{s}$ for $\mathbf{F}(x,y) = (x-y, y-x)$ and C the square bounded by $x=0, x=1, y=0, y=1$.
Solution: One could just parametrize the boundary of the square, but this involves four line segments, so it's better to use Green's theorem. Recall that Green's theorem says that if R is a region in $\mathbb{R}^2$, then

$$\int_{boundary\,of\,R} \mathbf{F}\cdot d\mathbf{s} = \int\int_R \frac{\partial}{\partial x}(\text{second component of }\mathbf{F}) - \frac{\partial}{\partial y}(\text{first component of }\mathbf{F})\,dA.$$

So here we get

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{x=0}^{x=1}\int_{y=0}^{y=1} -1-(-1)\,dA = 0.$$

2. Compute $\int_C \mathbf{F}(x,y)\cdot d\mathbf{s}$ for $\mathbf{F}(x,y) = (\tan^{-1}(\frac{y}{x}), \ln(x^2+y^2))$ and C the boundary of the region defined by the polar coordinate inequalities $1 \le r \le 2$, $0 \le \theta \le \pi$.
Solution: We again use Green's theorem, we get

$$\begin{aligned}\textstyle\int_C \mathbf{F}\cdot d\mathbf{s} &= \textstyle\int\int_R \frac{\partial}{\partial x}(\ln(x^2+y^2)) - \frac{\partial}{\partial y}(\tan^{-1}(\frac{y}{x}))\,dA = \\ &= \textstyle\int\int_R \frac{2x}{x^2+y^2} - \frac{1}{1+(\frac{y}{x})^2}\frac{1}{x}\,dA.\end{aligned}$$

We have to integrate over the region defined by the polar coordinate inequalities $1 \le r \le 2$, $0 \le theta \le \pi$. This suggests that we should switch to polar coordinates. The integral then becomes

$$\int_{r=1}^{r=2}\int_{\theta=0}^{\theta=\pi}\left[\frac{2r\cos(\theta)}{r^2} - \frac{1}{1+\tan(\theta)^2}\frac{1}{r\cos(\theta)}\right] r d\theta dr.$$

Note that we had to add an r–factor because we switched to polar coordinates. This simplifies to

$$\int_{r=1}^{r=2}\int_{\theta=0}^{\theta=\pi} 2\cos(\theta) - \frac{1}{\frac{\sin^2(\theta)+\cos^2(\theta)}{\cos^2(\theta)}}\frac{1}{\cos(\theta)} d\theta dr.$$

which becomes

$$\int_{r=1}^{r=2}\int_{\theta=0}^{\theta=\pi} 2\cos(\theta) - \cos(\theta) d\theta dr.$$