Math 355  
Exam #2  
October 5, 2004

This exam consists of 2 sections. You do not have to show any work for the problems in Part I. Solutions to problems in Part II, however, must show work to earn full credit. Pledge your exam before turning it in. Good luck!

**Part I:** No justification is necessary for the following questions.

1. Let \( A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \).
   
   (a) What is the dimension of the column space of \( A \)?
   
   (b) What is the dimension of the row space of \( A \)?

2. What is the definition of a linear transformation?

3. Let \( \varphi : V \to W \) be a linear transformation between finite dimensional vector spaces. What is the formula relating
   
   • the dimension of the kernel of \( \varphi \),
   
   • the dimension of the image space of \( \varphi \),
   
   • the dimension of \( V \).

4. Let \( \varphi_1, \varphi_2 : V \to W \) be linear transformations. Is the map \( \varphi : V \to W \) defined by \( \varphi(v) := \varphi_1(v) + \varphi_2(v) \) necessarily a linear transformation?

5. Let \( B = \{v_1; v_2; v_3\} \) be an ordered basis for a vector space \( V \). Determine \( c_B(v_1), c_B(0) \) and \( c_B(v_3 - v_2) \).

6. Let \( v_1, \ldots, v_k \) be linearly independent vectors in a \( k \)-dimensional vector space \( V \). Is \( \{v_1, \ldots, v_k\} \) necessarily a basis for \( V \)?

7. Let \( B := \{v_1, v_2\} \) be an ordered basis for \( V \) and let \( \tilde{B} := \{v_1 + v_2; v_2\} \). What is the matrix \( A \) such that \( c_B(v) = Ac_{\tilde{B}}(v) \) for all \( v \)?

**Part II on reverse**
Part II:

1. Consider

\[
\begin{align*}
  v_1 &:= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\
  v_2 &:= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \\
  v_3 &:= \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \\
  v_4 &:= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\end{align*}
\]

Among \(v_1, \ldots, v_4\) find a basis for \(\text{span}(v_1, \ldots, v_4)\).

2. Let \(V\) be the vector space of all polynomials of degree at most four. Determine whether \(f_1 := 1 - t, f_2 := 1 + 2t^2, f_3 := t + 2t^2\) are linearly independent or not.

3. Let \(V\) be the vector space of all polynomials of degree less or equal than two. Let \(B\) be the ordered basis \(\{1; t; t^2\}\). Consider the linear transformation \(\varphi : V \to V\) given by \(\varphi(p(t)) = p(2) \cdot (t + 2)\). Which matrix represents \(\varphi\) with respect to the basis \(B\)?

4. Let \(V\) be the vector space of all polynomials of degree less or equal than 27.

   (a) Show that the map \(\varphi : V \to \mathbb{R}\) given by \(\varphi(p(t)) := \int_0^1 p(t) \, dt\) is a linear transformation.

   (b) Show that \(\varphi\) is onto, i.e. show that the image space of \(\varphi\) is \(\mathbb{R}\).

   (c) Determine the dimension of the kernel of \(\varphi\).