Part I:

1. Let A, B, and C be nonsingular $p \times p$ matrices. Then

$$(ABC)(C^{-1}B^{-1}A^{-1}) = ABCC^{-1}B^{-1}A^{-1}$$

equals the identity matrix. Therefore $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. But in general $C^{-1}B^{-1}A^{-1} \neq A^{-1}B^{-1}C^{-1}$, so the answer is NO.

2. The answer is NO. For example take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = -A$. Then $\det(A) = \det(B) = 1$ but $\det(A+B) = \det \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$.

3. The answer is NO. The reason is that $A\mathbf{x} = \mathbf{b}$ does not necessarily have to have a solution at all. For example take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then the homogenous system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions but the system $A\mathbf{x} = \mathbf{b}$ has no solution.

4. The answer is NO. If A is a $p \times q$ matrix, then

$$AB = AC \Rightarrow B = C$$

for all $q \times r$ matrices B and C if and only if A has a left inverse. An extreme counter example would be to take $A = 0$ and B, C any choice of different $q \times r$–matrices.

5. Let $\mathbf{b}$ be any non–zero vector and $\mathbf{v}, \mathbf{w}$ be vectors that solve the system $A\mathbf{x} = \mathbf{b}$. Then $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \mathbf{b} + \mathbf{b}$, which is not equal to $\mathbf{b}$. So the answer is NO.

6. The answer is YES. For example use the first column at each step to compute the determinant.

7. The answer is YES. This can be seen using the third row to compute the determinants of A and B.

Part II:

1. The augmented matrix corresponding to the equation system is

$$\left(\begin{array}{ccc|c} 2 & 4 & -4 & 0 \\ 4 & 3 & -3 & -1 \end{array}\right)$$

This matrix can be turned into Gauss–Jordan form as follows:

$$\left(\begin{array}{ccc|c} 2 & 4 & -4 & 0 \\ 4 & 3 & -3 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 4 & 3 & -3 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & -5 & 5 & -1 \end{array}\right)$$
$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & -1 & \frac{1}{5} \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{2}{5} \\ 0 & 1 & -1 & \frac{1}{5} \end{array}\right)$$

A particular solution is therefore given by $\begin{pmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ 0 \end{pmatrix}$, the general solution for the homogeneous equation system is given by $\lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$. The general solution to the inhomogeneous equation system is therefore

$$\begin{pmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$$

2. Adding multiples of rows to other rows we can simplify the matrix without changing the determinant:

$$\det \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 \end{pmatrix} = \det \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix} = \det \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -2 & -1 \end{pmatrix}$$

Therefore $\det(A) = (-1)(-1)(2(-1) - 3(-2)) = 4$.

Note that the third column of B equals zero, hence if we use the third column to expand $\det(B)$, we see immediately that $\det(B) = 0$.

3. Recall that $U \subset \mathbb{R}^3$ is a subspace if and only if the following hold:

   (a) U is non–empty,

   (b) if $u, v \in U$, then $u + v \in U$,

   (c) if $u \in U, \lambda \in \mathbb{R}$, then $\lambda u \in U$.

   For easier writing we view vectors in $\mathbb{R}^3$ as 1×3–matrices. First consider

$$U := \left\{ \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \in \mathbb{R}^3 | v_1 - v_3 = 0 \right\}.$$

   (a) U is non–empty since for example $(0\ 0\ 0) \in U$.

   (b) Let $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}, \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \in U$, i.e. $v_1 - v_3 = 0$ and $w_1 - w_3 = 0$. Then

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{pmatrix}$$

   but this vector is in U since $(v_1 + w_1) - (v_3 + w_3) = v_1 - v_3 + w_1 - w_3 = 0$.

   (c) Let $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \in U$, i.e. $v_1 - v_3 = 0$ and $\lambda \in \mathbb{R}$. Then

$$\lambda \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} \lambda v_1 & \lambda v_2 & \lambda v_3 \end{pmatrix}$$

   but this vector is in U since $\lambda v_1 - \lambda v_3 = \lambda(v_1 - v_3) = 0$.

Therefore U is a subspace of $\mathbb{R}^3$.

Now consider

$$U := \left\{ \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \in \mathbb{R}^3 | v_1 v_2 = 0 \right\}.$$

(a) U is non–empty since for example $(0\ 0\ 0) \in U$.

(b) Let $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}, \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \in U$, i.e. $v_1 v_2 = 0$ and $w_1 w_2 = 0$. Then

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{pmatrix}$$

We compute $(v_1 + w_1)(v_2 + w_2) = v_1 v_2 + v_1 w_2 + w_1 v_2 + w_1 w_2 = v_1 w_2 + v_2 w_1$. This doesn't look like zero. But to show that (b) does not hold one has to give explicit examples for vectors in U where this term is not zero. For example take $v = (1\ 0\ 0)$ and $w = (0\ 1\ 0)$. Both vectors lie in U, but $v + w = (1\ 1\ 0)$ does not lie in U. Therefore U is not a subspace.

4. Consider the following augmented matrix, representing a linear system $A\mathbf{x} = \mathbf{b}$, in 5 unknowns:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 1 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{array} \right].$$

(a) This matrix is in row echelon form.

(b) A particular solution is given by $(1\ 2\ 0\ 2\ 0)$, the general solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is given by

$$\lambda_1(0\ \ -1\ 1\ 0\ 0\ ) + \lambda_2(-3\ 3\ 0\ 0\ 1), \lambda_1, \lambda_2 \in \mathbb{R}.$$

The general solution to the inhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is given by

$$(1\ 2\ 0\ 2\ 0) + \lambda_1(0\ \ -1\ 1\ 0\ 0\ ) + \lambda_2(-3\ 3\ 0\ 0\ 1), \lambda_1, \lambda_2 \in \mathbb{R}.$$

(c) The rank is the number of leading colums, so it equals 3.

5. Let A be a square nonsingular matrix, then

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

It follows that $(A^{-1})^T$ is the inverse to A, i.e. $(A^T)^{-1} = (A^{-1})^T$.