Math 355 **Exam #2** **Solution**

Part I: No justification is necessary for the following questions.

2. Let V be the vector space of all polynomials $p(t)$ of degree less or equal than 4 such that $p(3) = 0$.

   (a) What is $\dim(V)$?
   Solution: It's one less than the dimension of the vector space of all polynomials of degree less or equal than 4, i.e. one less than 5. So $\dim(V) = 4$.

   (b) Write down a basis for V.
   A possible answer is $t-3, (t-3)^2, (t-3)^3, (t-3)^4$.

3. Let $v_1, \ldots, v_k$ be linearly independent vectors in a vector space V. Let w be the zero vector. Are $w, v_1, \ldots, v_k$ linearly independent?
   Solution: No, for example $2w + 0v_1 + \cdots + 0v_k$ is a linear combination which equals 0, but not all coefficients are zero.

4. Let $v_1, \ldots, v_k$ be linearly dependent vectors in a vector space V. Assume that $w \in \operatorname{span}\{v_1, \ldots, v_k\}$. How many solutions does the equation $\lambda_1 v_1 + \cdots + \lambda_k v_k = w$ have, i.e. how many such λ_i's can we find? (possible answers are: none, exactly one, 23, infinitely many,...)
   Solution: There are infinitely many solutions to $\lambda_1 v_1 + \cdots + \lambda_k v_k = 0$. Furthermore the equation $\lambda_1 v_1 + \cdots + \lambda_k v_k = w$ has at least one solution since $w \in \operatorname{span}\{v_1, \ldots, v_k\}$. Since we can add any solution of $\lambda_1 v_1 + \cdots + \lambda_k v_k = 0$ to a solution of $\lambda_1 v_1 + \cdots + \lambda_k v_k = w$ we see that $\lambda_1 v_1 + \cdots + \lambda_k v_k = w$ also has infinitely many solutions.

5. Let $\varphi : V \to W$ be a linear transformation which is onto. Assume that $\dim(V) = k$, $\dim(W) = l$. What is $\dim(\operatorname{Ker}(\phi))$?
   Solution: We know that
   $$\dim(V) = \dim(\operatorname{Im}(\varphi)) + \dim(\operatorname{Ker}(\phi)).$$
   But $\operatorname{Im}(\varphi) = W$, so it follows that $\dim(\operatorname{Ker}(\phi)) = \dim(V) - \dim(V) = l - k$.

6. Let $B = \{v_1; v_2; v_3\}$ be an ordered basis for a vector space V. Assume that $w \in V$ is a vector such that $c_B(w) = \begin{pmatrix} 1 & -2 & 3 \end{pmatrix}^t$. Let C be the ordered basis $\{v_3; v_1 - v_2; v_2\}$. What is $c_C(w)$?
   Solution: Since $c_B(w) = \begin{pmatrix} 1 & -2 & 3 \end{pmatrix}^t$ it follows that $w = v_1 - 2v_2 + 3v_3$, which we now have to rewrite in terms of C. We get
   $$w = v_1 - 2v_2 + 3v_3 = 3v_3 + (v_1 - v_2) - v_2.$$
   Therefore $c_C(w) = (3, 1, -1)$.

Part II on reverse

Part II:

1. Consider

$$v_1 := \begin{pmatrix}1\\0\\2\\1\end{pmatrix}, v_2 := \begin{pmatrix}0\\3\\1\\0\end{pmatrix}, v_3 := \begin{pmatrix}2\\6\\4\\2\end{pmatrix}.$$

   (a) Show that v_1, v_2, v_3 are linearly independent.
   Solution: Put v_1, v_2, v_3 into a big matrix and compute the row echelon form. You get three leading columns, which means that the original vectors are linearly independent.

   (b) How many vectors do you have to add to v_1, v_2, v_3 to get a basis for $\mathbb{R}^4$?
   Solution: Since a basis for $\mathbb{R}^4$ has to have 4 vectors we have to add one vector.

   (c) Find vectors $w_1, \ldots, w_l$ (note that l is your answer from (b)) such that $v_1, v_2, v_3, w_1, \ldots, w_l$ form a basis for $\mathbb{R}^4$.
   Solution: Take the vectors $v_1, v_2, v_3, e_1, e_2, e_3, e_4$ and put them into a big matrix. Compute the row echelon form. Then vectors among $v_1, v_2, v_3, e_1, e_2, e_3, e_4$ which correspond to the leading columns form a basis. In this case you will see that v_1, v_2, v_3, e_1 are a basis.

2. Let V be the vector space of all polynomials of degree at most three.

   (a) Determine whether

$$p_1(t) := 1 - t, p_2(t) := -1 + 2t + 3t^2 + t^3, p_3(t) := t^2 + 5t^3, p_4(t) = -1 + 3t + 7t^2 + 7t^3$$

   are linearly independent or not.
   Solution: Pick an ordered basis for V, e.g. $B = \{1; t; t^2; t^3\}$. Then we only have to check whether

$$c_B(p_1(t)) = \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, c_B(p_2(t)) = \begin{pmatrix}-1\\2\\3\\1\end{pmatrix}, c_B(p_3(t)) = \begin{pmatrix}0\\0\\1\\5\end{pmatrix}, c_B(p_4(t)) = \begin{pmatrix}-1\\3\\7\\7\end{pmatrix}$$

   are linearly independent. We consider the big 4×4–matrix with columns given by the above vectors, after computing the row echelon form you will see that the last column is a non–leading column. So the vectors are linearly dependent.

   (b) Find $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that

$$\lambda_1 p_1(t) + \lambda_2 p_2(t) + \lambda_3 p_3(t) + \lambda_4 p_4(t) = -3 + 8t + 18t^2 + 20t^3.$$

   (you do NOT have to find ALL such λ's).
   Solution: Again using the ordered basis $B = \{1; t; t^2; t^3\}$ this boils down to finding a solution to

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & -1 & -3 \\ -1 & 2 & 0 & 3 & 8 \\ 0 & 3 & 1 & 7 & 18 \\ 0 & 1 & 5 & 7 & 20 \end{array}\right)$$

3. Let V be the vector space of all polynomials of degree at most three with ordered basis $B = \{1; t-1; t^2-1; t^3-1\}$. Furthermore let $W = \mathbb{R}^4$ with the ordered basis $\{2e_1; 3e_2; 4e_3; 5e_4\}$ (recall that $e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^t$ and so on). Now consider the linear transformation $\varphi : V \to W$ given by

$$\varphi(p(t)) = \begin{pmatrix} p(1) \\ p(2) \\ p''(1) \\ p''(2) - p(0) \end{pmatrix}$$

   (a) Find the matrix which represents φ with respect to the ordered bases B and C.
   Solution: Write $C = \{2e_1; 3e_2; 4e_3; 5e_4\}$. First note that

$$c_C \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = c_C(ae_1+be_2+ce_3+de_4) = c_C\left(\frac{1}{2}a \cdot 2e_1 + \frac{1}{3}a \cdot 3e_2 + \frac{1}{4}a \cdot 4e_3 + \frac{1}{5}a \cdot 5e_4\right) = \begin{pmatrix} \frac{1}{2}a \\ \frac{1}{3}b \\ \frac{1}{4}c \\ \frac{1}{5}d \end{pmatrix}.$$

   We now compute

$$c_C(\varphi(1)) = c_C \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ 0 \\ -\frac{1}{5} \end{pmatrix}$$

$$c_C(\varphi(t-1)) = c_C \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{3} \\ 0 \\ \frac{1}{5} \end{pmatrix}$$

$$c_C(\varphi(t^2-1)) = c_C \begin{pmatrix} 0 \\ 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \\ \frac{3}{5} \end{pmatrix}$$

$$c_C(\varphi(t^3-1)) = c_C \begin{pmatrix} 0 \\ 7 \\ 6 \\ 13 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{7}{3} \\ \frac{3}{2} \\ \frac{13}{5} \end{pmatrix}$$

   So the matrix representing φ is

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & \frac{7}{3} \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{5} & \frac{1}{5} & \frac{3}{5} & \frac{13}{5} \end{pmatrix}$$

   (b) Is φ invertible? Justify your answer.
   Solution: φ is invertible if the matrix representing φ is invertible. But an easy computation shows that the determinant of the above matrix is non–zero.

4. Let V be the vector space of all functions defined on $\mathbb{R}$. Define $\varphi : V \to V$ by $\varphi(f(x)) := f(x-2)$.

(a) Show that φ is a linear transformation.
Solution: So let $f(x), g(x) \in V$. Then

$$\varphi(f(x) + g(x)) = f(x-2) + g(x-2) = \varphi(f(x)) + \varphi(g(x))$$

similarly for scalar multiplication.

(b) What is the inverse map, i.e. what is the linear transformation $\psi : V \to V$ such that $\psi(\varphi(f(x)) = f(x)$ and $\varphi(\psi(f(x)) = f(x)$ for all functions $f(x)$?
Solution: Define $\psi(f(x)) := f(x+2)$. Then $\psi(\varphi(f(x))) = \psi(f(x-2)) = f((x+2)-2) = f(x)$, and $\varphi(\psi(f(x))) = \varphi(f(x+2)) = f((x-2)+2) = f(x)$